THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

2 BUXTON AVENUE, CAVERSHAM, READING

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XVII. DECEMBER, 1933. No. 226.

MATHEMATICS AT THE BRITISH ASSOCIATION, 1933.

BY J. H. C. WHITEHEAD.

THE mathematical section had its own discussions on the mornings of 7th and 8th September, in addition to which there was a discussion on the 12th held in conjunction with the psychology section.

The morning of the 7th was devoted to analysis, the speakers being Dr. E. H. Linfoot, Dr. L. S. Bosanquet and Dr. A. C. Offord. Linfoot gave an interesting account of certain problems in the theory of numbers which arise in connection with the representation of a number as a sum of numbers of specified types; Bosanquet talked about the "summability" of Fourier series* and Offord about Fourier transforms. In this kind of mathematics the interest is largely in the details of proof, and it is not easy to give a short talk which is to be of interest to non-specialists. However, the speakers rose to the occasion and gave the audience a clear idea of what they were after.

On 8th September there were four papers read, each on geometry. The first was an account† by Mr. W. V. D. Hodge of the brilliant work he has been doing on multiple integrals attached to an algebraic variety. This work, as yet partly unpublished, is likely to prove as exciting as anything which has been done in geometry for some time. He was followed by Dr. D. W. Babbage on the representation of Cremona transformations by transformations, such as projections, which are capable of a geometrical construction. Dr. P. Du Val and Mr. J. H. C. Whitehead read papers on multiple planes, with special reference to branch curves and their singular points, and on the

* *See* pp. 300-302 of this number of the *Gazette*.

† *See* pp. 303-306 of this number of the *Gazette*.

calculus of variations, respectively. The final paper was by Mr. H. G. Green on the generalisation of Pascal's theorem to n dimensions.

The joint discussion on Tuesday was on " The validity and value of methods of correlation ". The speakers on the paper were Prof. C. Spearman, Dr. W. Brown, Dr. J. Wishart, Dr. S. Dawson, Dr. S. S. Wilks, Dr. J. O. Irwin, and Prof. H. T. H. Piaggio, and Prof. R. A. Fisher was present to indicate tactfully, as he said later in conversation, that the psychologists were being sold a pup. Unfortunately the discussion clashed with the meeting held by Section A, on the expanding universe, and I attended the latter. After our domestic goings-on in Section A* it was impressive to find a large hall quite full, with people standing three deep round the door. The papers read were of first-class scientific interest, but were so well prepared as to be partially intelligible to people with little or no mathematical training. The speakers were Sir A. S. Eddington, Prof. E. A. Milne, Dr. G. C. McVittie, Dr. W. H. McCrea, M. l'Abbé Lemaitre and Prof. W. de Sitter. Prof. Milne gave a brilliant account of his representation of the universe—described by the more orthodox McVittie as " Professor Milne's stimulating heresies ". It is amusing to find Milne using " creation " as a technical term for a " singularity in the past history of an observer ".

Finally, a word of thanks and congratulation to Dr. Dorothy Wrinch for her successful efforts in organising Section A*. Perhaps even more encouraging than the success of the 1933 meeting is her determination that the meeting next year at Aberdeen shall be still more interesting.

J. H. C. W.

FOURTH INTERNATIONAL CONGRESS FOR APPLIED MECHANICS.

THE fourth International Congress for Applied Mechanics will be held in Cambridge, during the period 3rd-9th July, 1934. The organising committee consists of L. Bairstow, W. S. Farren, C. E. Inglis, B. Melvill Jones, R. V. Southwell and G. I. Taylor.

Subjects for discussion are to be grouped under the following general heads: (i) Rational Mechanics; (ii) Mechanics of Fluids; (iii) Materials; (iv) Water Waves. In order to avoid the considerable expense of publishing complete *Proceedings*, only summaries of papers and short abstracts of discussions will appear in the printed volume.

A fee of 15 shillings will be payable by all taking part in the scientific or social activities of the Congress. For further information, including details of the arrangements and particulars of accommodation available, application should be made to The Secretary, Fourth International Congress for Applied Mechanics, Engineering Laboratory, Cambridge.

THE ENVELOPE OF THE SIMSON LINES OF A TRIANGLE.

By A. Lodge.

In a triangle ABC (Fig. 1) let O be the circumcentre, OD the perpendicular on BC; let H be the orthocentre, and U the mid-point of AH,

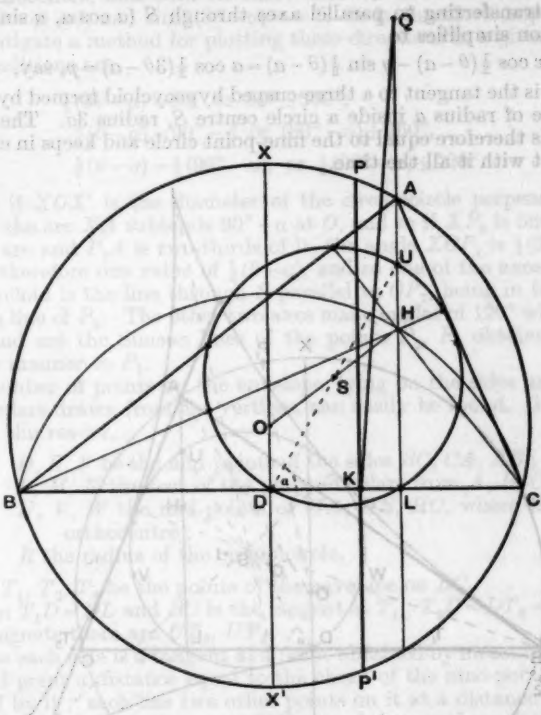


Fig. 1.

so that $AU = UH = OD$. Let S bisect OH, so that S is the centre of the nine-point circle, and UD a diameter, being equal and parallel to OA. Suppose the angle $SDC = \alpha$, and $SD = a$. Take P any point on the circumcircle, and denote by θ the angle OP makes with BC, and draw the chord PP' perpendicular to BC, cutting it in K. The Simson line of P passes through K and is parallel to $P'A$, so that it cuts AH in Q, such that $QA = KP'$.

Thus
$$\angle PKQ = \angle PP'A = \tfrac{1}{2}\angle POA$$
$$= \tfrac{1}{2}(\theta - \alpha),$$

and the perpendicular from D on KQ

$$= DK \cos \tfrac{1}{2}(\theta - \alpha)$$
$$= 2a \cos \theta \cos \tfrac{1}{2}(\theta - \alpha)$$
$$= a\{\cos \tfrac{1}{2}(3\theta - \alpha) + \cos \tfrac{1}{2}(\theta + \alpha)\}.$$

Thus the equation of KQ, taking BC, DO as axes, is

$$x \cos \tfrac{1}{2}(\theta - \alpha) - y \sin \tfrac{1}{2}(\theta - \alpha) = a\{\cos \tfrac{1}{2}(3\theta - \alpha) + \cos \tfrac{1}{2}(\theta + \alpha)\}.$$

On transferring to parallel axes through S ($a \cos \alpha$, $a \sin \alpha$), the equation simplifies to

$$x \cos \tfrac{1}{2}(\theta - \alpha) - y \sin \tfrac{1}{2}(\theta - \alpha) = a \cos \tfrac{1}{2}(3\theta - \alpha) = p, \text{ say, } \ldots\ldots(1)$$

which is the tangent to a three-cusped hypocycloid formed by rolling a circle of radius a inside a circle centre S, radius $3a$. The rolling circle is therefore equal to the nine-point circle and keeps in external contact with it all the time.

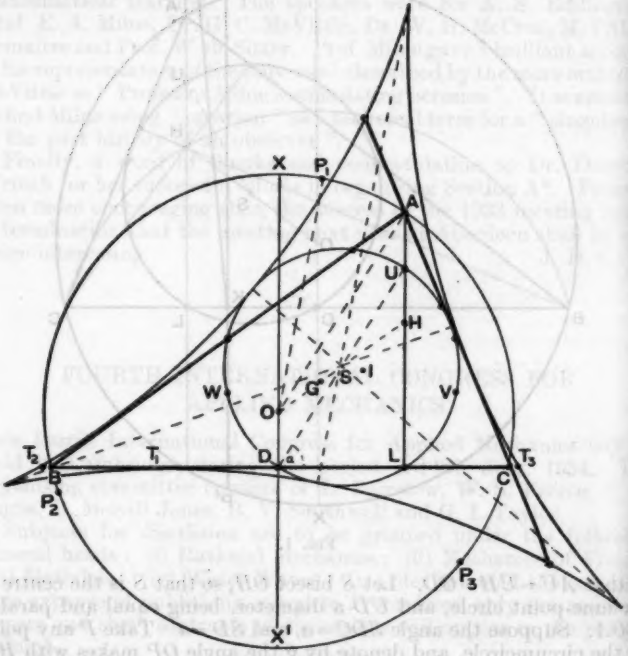


FIG. 2.

The normal to the envelope of (1) is

$$x \sin \tfrac{1}{2}(\theta - \alpha) + y \cos \tfrac{1}{2}(\theta - \alpha) = 3a \sin \tfrac{1}{2}(3\theta - \alpha), \ldots\ldots(2)$$

so, squaring and adding, the pedal equation of the envelope is

$$r^2 = p^2 + 9(a^2 - p^2), \ldots\ldots(3)$$

showing that the point of contact on (1) lies outside the nine-point circle at a distance from it equal to its chord cut off on (1), since half this chord is $\sqrt{(a^2-p^2)}=\frac{1}{3}\sqrt{(r^2-p^2)}$. The radius vector has its maximum value $3a$ when $p=0$, that is, when $\cos\frac{1}{2}(3\theta-\alpha)=0$.

It is remarkable that, for all triangles inscribed in a given circle, the envelope is the same symmetrical three-cusped hypocycloid, merely shifted so that its centre lies half-way between circumcentre and orthocentre, and with its three axes oriented so as to make $\cos\frac{1}{2}(3\theta-\alpha)=0$, and $\sin\frac{1}{2}(3\theta-\alpha)=+1$. It is, therefore, important to investigate a method for plotting these directions in a given case. The conditions are

$$\tfrac{1}{2}(3\theta-\alpha)=90^\circ \text{ or } 90^\circ\pm360^\circ,$$

or
$$\tfrac{3}{2}(\theta-\alpha)=90^\circ-\alpha \text{ or } (90^\circ-\alpha)\pm360^\circ,$$

or
$$\tfrac{1}{2}(\theta-\alpha)=\tfrac{1}{3}(90^\circ-\alpha), \text{ or } \tfrac{1}{3}(90^\circ-\alpha)\pm120^\circ.$$

Now if XOX' is the diameter of the circumcircle perpendicular to BC, the arc XA subtends $90^\circ-\alpha$ at O, and so if XP_1 is one-third of this arc and P_1A is two-thirds of it, the angle XOP_1 is $\frac{1}{3}(90^\circ-\alpha)$ and is therefore one value of $\frac{1}{2}(\theta-\alpha)$, and so one of the axes of the hypocycloid is the line through S parallel to OP_1, being in fact the Simson line of P_1. The other two axes make angles of 120° with this axis, and are the Simson lines of the points P_2, P_3 obtained in a similar manner to P_1.

A number of points on the envelope, lying on the sides and perpendiculars drawn from the vertices, can easily be found. Proof is left to the reader.

Let D, E, F be the mid-points of the sides BC, CA, AB;
L, M, N the feet of the perpendiculars from A, B, C;
U, V, W the mid-points of HA, HB, HC, where H is the orthocentre;
R the radius of the circumcircle.

Let T_1, T_2, T_3 be the points of the envelope on BC.
Then $T_1D=DL$ and BC is the tangent at T_1; $T_2D=DT_3=R$, and the tangents there are UT_2, UT_3.

Thus each side is a tangent at a point obtained by measuring from its mid-point a distance equal to the chord of the nine-points circle cut off by it; each has two other points on it at a distance R from its mid-point, and the tangents at these points pass through the mid-point of the opposite pair of the four points A, B, C, H. The extreme points on AH are at a distance R from U, and the tangents at these points pass through D. This gives 18 points on the envelope and the tangents at these points.

Further, such a line as UT_2 crosses the nine-points circle at its mid-point and if from that point we measure distances R along UT_2 in both directions, we get the extreme points of this chord, and the same with all the other auxiliary tangents.

There are two from each of U, V, W, D, E, F, so we get 24 points, giving 42 in all.

A. L.

A PROOF OF THE BINOMIAL THEOREM.

By S. Barnard.

In teaching algebra, it is obviously desirable to prove the Binomial Theorem for a real (or at any rate for a rational) index without assuming certain theorems on convergence which are used in Euler's proof. So far as I know, no one has thought it worth while to devise such a proof.

The following depends on the fact that if n is real and $x<0$, the derivative of x^n exists and is equal to nx^{n-1}.

To show that *if n and x are real and $|x|<1$, then*

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots.$$

The case in which n is a positive integer is ignored. Let $t_{n,r}$ be the rth term and $S_{n,r}$ the sum to r terms of the series. Also let

$$y_{n,r}=(1+x)^n-S_{n,r}.$$

First suppose that $0<x<1$, then

$$y_{n,1}=(1+x)^n-1\gtrless 0 \text{ as } n\gtrless 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also $$\frac{dy_{n,r}}{dx}=ny_{n-1,r-1} \text{ and } y_{n,r}=0 \text{ when } x=0. \quad \ldots\ldots\ldots\ldots(2)$$

If for given values of n and r it can be shown that $y_{n-1,r-1}$ is of invariable sign for $0<x<1$, then by (2), $y_{n,r}$ must have the same sign as $y_{n-1,r-1}$.

Repeating this reasoning when $n-1, n-2, \ldots$ are substituted for n and observing that by (1), $y_{n-r+1,1}$ has the same sign as $n-r+1$, it follows that the expressions

$$y_{n,r}, n(n-1)(n-2)\ldots(n-r+2)y_{n-r+1,1} \text{ and } n(n-1)(n-2)\ldots(n-r+1)$$

all have the same sign. Therefore

$y_{n,r}$ has the same sign as $t_{n,r+1}$.

After a certain stage, say for r greater than a fixed number k, the signs of $t_{n,r+1}$ and consequently those of $y_{n,r}$ are alternately positive and negative, as r increases.

Therefore $(1+x)^n$ lies between $S_{n,r}$ and $S_{n,r+1}$ for $r>k$. But since the series is convergent, $S_{n,r+1}-S_{n,r}\to 0$ as $r\to\infty$. Hence

$$S_{n,r}\to(1+x)^n,$$

which proves this case of the theorem.

Next let $x<0$. Changing the sign of x, it has to be shown that *if* $0<x<1$, *then*

$$(1-x)^n=1-nx+\frac{n(n-1)}{2!}x^2-\ldots+(-1)^r\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots.$$

In this case $y_{n,r}=(1-x)^n-S_{n,r}$, so that

$$y_{n,1}=(1-x)^n-1 \gtrless 0 \text{ as } n \gtrless 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

also $$\frac{dy_{n,r}}{dx}=-ny_{n-1,r-1} \text{ and } y_{n,r}=0 \text{ when } x=0. \quad \ldots\ldots\ldots(4)$$

Reasoning as in the previous case and observing that by (3) y_{n-r+1} and $n-r+1$ have opposite signs, it will be seen that the expressions

$$y_{n,r},\ (-1)^{r-1}\,n(n-1)\ldots(n-r+2)y_{n-r+1,1}$$

and

$$(-1)^r\,n(n-1)\ldots(n-r+1)$$

all have the same signs. Hence as before

$$y_{n,r} \textit{ has the same sign as } t_{n,r+1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

After a certain stage, say for $r>k$, the sign of $t_{n,r+1}$ is invariable and we proceed as follows.

Let $$z_{n,r}=y_{n,r}-t_{n,r+1}(1-x)^{-1}.$$

It is easily seen that

$$\begin{aligned}(1-x)z_{n,r}&=(1-x)y_{n,r}-t_{n,r+1}\\&=y_{n+1,r}-\frac{n+1}{n-r+1}\,t_{n,r+1}\\&=y_{n+1,r}-t_{n+1,r+1}\\&=y_{n+1,r+1}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)\end{aligned}$$

Hence by (5), $z_{n,r}$ has the same sign as $t_{n+1,r+2}$.

Now $$t_{n+1,r+2}=-\frac{n+1}{r+2}\,x\,.\,t_{n,r+1}, \text{ hence by (5),}$$

$$z_{n,r} \textit{ has the same sign as } -(n+1)y_{n,r}.$$

(i) *If* $n>-1$, it follows that $y_{n,r}$ and $z_{n,r}$ have opposite signs. Therefore $y_{n,r}$ lies between 0 and $t_{n,r+1}\,.\,(1-x)^{-1}$. But $t_{n,r+1}\to 0$ as $r\to\infty$, therefore $y_{n,r}\to 0$ and $s_{n,r}\to(1-x)^n$.

(ii) *If* $n>-1$, $y_{n,r}$ and $z_{n,r}$ have the same sign and we proceed thus:

If it can be shown that $y_{n+1,r}\to 0$ as $r\to\infty$, then $y_{n+1,r+1}\to 0$ and by (6), $z_{n,r}\to 0$ and consequently $y_{n,r}\to 0$.

Let $-m-1<n<-m$, where m is a positive integer; then $n+m>-1$ and by case (i) $y_{n+m}\to 0$. Hence in succession

$$y_{n+m-1,r},\ y_{n+m-2,r},\ \ldots\ y_{n,r}$$

tend to zero and so $s_{n,r}\to(1-x)^n$. Thus the theorem is proved in every case. S.B.

GLEANINGS FAR AND NEAR.

946. The state as a system of legal imperatives is a temporary parallelogram of forces, the character of which shifts as the forces alter which determine its momentary position.—H. J. Laski, *Introduction to Politics*. [Per Mr. B. B. Bagi.]

GAMES OF CHANCE.

I. THE BANKER'S CLOCK.*

By S. T. Shovelton.

The game of Banker's Clock provides an interesting question in mathematical probability. In this game the banker turns up in sequence the first twelve cards of a well-shuffled ordinary pack of 52 cards. He backs himself to turn up at least one card of which the face value corresponds to its position in the sequence, an Ace ranking as one, a Jack as eleven and a Queen as twelve. The interest in the question from the mathematical point of view is in finding the probability that the event will happen.

Consider the number of ways of failure in the 52 !/40 ! different arrangements.

If $x_1, x_2, x_3, \ldots x_{13}$ denote the cards Ace, 2, 3, ... King, then the various arrangements which imply failure are the terms in the product

$$(x_2+x_3+\ldots+x_{13})(x_1+x_3+\ldots+x_{13})\ldots(x_1+x_2+\ldots+x_{11}+x_{13})$$

where there are 12 factors, $x_1, x_2, \ldots x_{12}$ being omitted in turn from the complete set $x_1+x_2+\ldots+x_{13}$. The number of ways for the arrangement represented by any one term in the product is the coefficient of that term multiplied by 4, 4 . 3, 4 . 3 . 2, or 4 . 3 . 2 . 1 in respect of each factor of the term according as that factor has the power 1, 2, 3 or 4. The reason for this is that x_a^n represents n drawings of a card of number a. Now there are originally 4 of any one number in the pack, and until one is drawn there are 4 ways of drawing that number, but when one is drawn the number is reduced to 3, and so on.

We have therefore to find the sum of the coefficients, appropriately weighted, of all the terms in

$$\left(\sum_1^{13} x_a\right)^{12} - \left(\sum_1^{12} x_r\right)\left(\sum_1^{13} x_a\right)^{11} + \left(\sum x_r x_s\right)\left(\sum_1^{13} x_a\right)^{10} - \left(\sum x_r x_s x_t\right)\left(\sum_1^{13} x_a\right)^{9} + \ldots + x_1 x_2 x_3 \ldots x_{12},$$

such that no one factor in any term is higher than the fourth power, where $r, s, t, \ldots$ range from 1 to 12 but no two of them are equal when associated together.

The appropriate terms in the expansion of $(x_1+x_2+\ldots+x_{13})^{12}$ with weighted coefficients are

$$\Sigma x_a^m x_b^n x_c^p \ldots \times \frac{12!}{m!\ n!\ p!\ldots} \times 4 \,.\, 3 \ldots (5-m) \times 4 \,.\, 3 \ldots (5-n) \times \ldots$$

where $m+n+p+\ldots=12$, and no one of $m, n, p, \ldots$ is greater than 4. This is equal to

$$12!\ \Sigma {}^4C_m x_a^m \,.\, {}^4C_n x_b^n \,.\, {}^4C_p x_c^p \ldots .$$

* An alternative proof, developed from the knowledge of the form of the result, is given in *Caliban's Problem Book*, recently published.

Hence the sum of the coefficients is $12\,!$ × the coefficient of x^{12} in

$$(1+4x+{}^4C_2x^2+\ldots+x^4)^{13}=52\,!/40\,!.$$

Now consider any term of the type $x_r(x_1+x_2+\ldots+x_{13})^{11}$. The appropriate terms, with weighted coefficients, are

$$4\Sigma x_a^m x_b^n x_c^p \ldots x_r^s \ldots \times \frac{11\,!}{m\,!\,n\,!\,p\,!\ldots} \times 4\,.\,3\ldots(5-m)\times 4\,.\,3\ldots(5-n)\times\ldots,$$

where $m+n+p+\ldots=11$, s is not greater than 3 and no other index is greater than 4, and where x_r^s appears the corresponding factor in the numerator is 3, 3 . 2, or 3 . 2 . 1 according as s is 1, 2 or 3. The sum of the coefficients is therefore $4\,.\,11\,!$ × the coefficient of x^{11} in

$$(1+4x+\ldots+x^4)^{12}(1+3x+3x^2+x^3)=4\,.\,(51\,!)/40\,!.$$

There are 12 such terms, giving a total coefficient $4\,.\,12\,.\,(51\,!)/40\,!$. The required coefficient obtained from $(\Sigma x_r x_s)(x_1+\ldots+_{13})^{10}$ can now be seen to be $4^2\,.\,{}^{12}C_2\,.\,(50\,!)/40\,!$, and the last coefficient is 4^{12}.

Thus, remembering that the total number of ways of failure and success is $52\,!/40\,!$, the chance of failure is

$$1-4\,.\,\frac{12}{52}+4^2\,.\,{}^{12}C_2\,.\,\frac{1}{52\,.\,51}-4^3\,.\,{}^{12}C_3\,\frac{1}{52\,.\,51\,.\,50}+\ldots+4^{12}\,.\,\frac{40\,!}{52\,!}.$$

The chance of the clock "striking" at least once is accordingly ·614 approximately, or the odds in favour of the banker are about 3 to 2.

S. T. Shovelton.

APPLICATION OF STATISTICS TO TECHNICAL PROBLEMS.

The researches in mathematical statistics which have been a feature in the development of twentieth-century science, and which have given the subject the right to rank as a not unimportant branch of applied mathematics, have usually been stimulated in the first place by the practical needs of the experimenter, in the biological and other fields of enquiry. Specific problems have in due course produced the technical processes necessary for their solution, yet methods developed in one branch may remain unknown in other branches, even where the same general solution is applicable. In particular we have in industry problems of sampling and routine testing where known results in pure research can be applied with profit, and where in turn new problems arise.

The Royal Statistical Society in this, its centenary, year has announced the formation of a special section for the purpose of promoting the application of methods of statistical analysis to industrial and agricultural problems. The time is considered ripe for the formal provision of facilities giving a common platform for discussion to those who have need of these methods in their work. Regular meetings are to be held, and a supplement to the Society's journal will be published, devoted entirely to this aspect of statistical science. At the first meeting on Nov. 23 Dr. R. H. Pickard, F.R.S., Director of the British Cotton Industry Research Association, dealt with the application of statistical methods to problems of production and research in industry. Further meetings are promised for Jan. 25, when Dr. J. Wishart will speak on "Statistics in Agricultural Research", and March 8, to be addressed by Dr. E. S. Pearson on "Sampling Problems in Mass Production Industry".

In a circular announcing its purpose, the Society asks that those interested communicate with the Hon. Secy. at 9 Adelphi Terr., W.C. 2, when they will be kept informed of developments.

THE QUESTION OF THE MOMENTUM.

By G. H. Grattan-Guinness.

". . . And so the impulse of the blow is equal to the change of momentum of the body."

This phrase, coming at the conclusion of a more or less standard lesson in Elementary Mechanics, must surely have struck many teachers with a feeling of shame for having participated in a piece of not entirely straightforward work, and their pupils with an idea that they have witnessed some downright "wangling". The usual impulse-momentum treatment cannot but be unsatisfactory to all concerned in the lesson, for it follows on the introduction of a new quantity of measurement of which, as so many of the text-books plaintively remark, "no special name has been given to the unit". If it is of so little apparent importance that we do not even trouble to name its units, why bother with it? And why, in heaven's name, measure "the effect of a blow" by the product of the force of the blow and some measurement of *time*?—the last thing that a beginner would expect, after some preliminary treatment of static and other forces measured in the usual way. Why bring in *time*?

The general method seems to be to make a bald statement that Momentum is a quantity which is measured by the product of mass and velocity (or, better, perhaps, as $W \cdot v/g$, from the fundamental equation $P/W = f/g$, as advised in the Association's Report on the Teaching of Mechanics), and, later, to define Impulse in terms of the product of the resisting force and the time of duration of the impact. It is easy, of course, to demonstrate the equivalence of these two quantities, but it seems without reason to have introduced either of them. One school text-book says: "The dynamical relations of a moving body are usually expressed conveniently in terms of momentum, by which we mean the product of the mass of the body and its velocity. . . . The mass of a body is, generally speaking, quite unchangeable, and changes in momentum are simply due to changes in velocity". No explanation of the convenience is given, and the pupil is told, in other words: "We are hardly likely to require this quantity; we could just as well do with the velocity we already know, but it is at least something else to worry you and for you to learn"—which is a lamentable state of affairs.

I suggest the following order of treatment of principles as permitting the introduction of momentum and impulse with some reasonable excuse:

After the usual elementary consideration of constant acceleration problems, proceed to Newton's Laws of Motion, and in particular to some experiment such as with Fletcher's Trolley or Atwood's Machine, obtaining the fundamental equation, $P/W = f/g$, and defining the second law as: "The weight of the body remaining constant, the acceleration produced is proportional to the impressed force, and takes place in the direction of the force". (I avoid the use of the word "mass" in the above for beginners—indeed, for some con-

siderable time; this, however, is, of course, a matter of choice.) Work and energy may here be conveniently considered.

Take then Hicks' Ballistic Pendulum or some similar apparatus, and perform the usual experiment to cause complete cessation of motion after impact of the two bodies concerned. If the bodies have weights W_1 and W_2 units, and meet with velocities v_1 and v_2 units, respectively, then, if complete loss of velocity takes place in a short interval of time t units, the second law tells us that, since the weight of W_1 remains constant, the average retardation of W_1 is v_1/g units, and the average resistance (probably practically constant in so short an interval of time) offered by W_2 to W_1 is given by

$$R = \frac{W_1}{g} \cdot \frac{v_1}{t}.$$

(So, of course, in the opposite direction, we have

$$R = \frac{W_2}{g} \cdot \frac{v_2}{t},$$

whence, as verified by the experiment,

$$W_1 \cdot \frac{v_1}{g} = W_2 \cdot \frac{v_2}{g},$$

or $M_1 v_1 = M_2 v_2$ if "masses" are employed—the foundation for the principle of Conservation of Linear Momentum.)

Now, we may say, in so short an interval of time it is very difficult to measure the sudden resistance R units, and it is equally difficult to measure the interval t units itself. But we have the relation: $Rt = W_1 \cdot v_1/g$; *i.e.* R and t, though themselves more or less immeasureable, can be measured together in product. For convenience, therefore (and purely as a matter of convenience), we shall measure "the effect of a blow" such as this, not by the magnitude of the resisting force itself ("Action and reaction are equal and opposite"), but by the product of the force and the short interval of time during which we may take it to be acting. This product is equivalent to the quantity $W \cdot v/g$ or Mv of the body causing the blow, or, more generally, to the change in this quantity, viz. $W(v-u)/g$ or $M(v-u)$. The quantity $W \cdot v/g$ or Mv we shall call the "momentum" of the body, and the quantity Rt the "impulse" of the blow. The second law may now be restated, if desired, in the more general form: "The rate of change of momentum is proportional to the impressed force, and takes place in the direction of the force".

This treatment, I suggest, is orderly in its reasoning and reasonable in its order. There are fair excuses for introducing the new quantity "momentum" and for including the time dimension in the measurement of "impulsive" forces which are not apparent under the scheme of direct definitions by products. Especially, by the way, would I urge the avoidance of that rather ambiguous phrase (ambiguous to beginners at least), "the quantity of motion" of a body.

G. H. G.-G.

THE ABSOLUTE SUMMABILITY OF FOURIER SERIES.*

BY L. S. BOSANQUET.

THE problem under discussion is the relation between the behaviour of a function $\phi(t)$ near a particular point $t=x$ and the behaviour of the partial sums s_n of its Fourier series at that point. We start from Hardy and Littlewood's theorem † that if $\phi(t)$ *tends to a limit in the Cesàro sense as* $t \to x$, *then the Fourier series at that point is summable in the Cesàro sense, and conversely.*‡ The exact order of the Cesàro means in each case is unspecified. Expressing this in a more general form, we have to compare a transform of the function,

$$T(t) = \int_0^t K_1(t, u)\, \phi(u)\, du, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and a transform of the series,

$$U_n = \sum_1^n K_2(n, \nu)\, s_\nu, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

or, if we prefer to replace n by a continuous variable,

$$V(\omega) = \int_0^\omega K_3(\omega, u)\, s(u)\, du, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where $$s(u) = \sum_{\nu \leqslant u} a_\nu.$$

If we take

$$K_1 = \kappa t^{-\kappa}(t-u)^{\kappa-1}, \quad (\kappa > 0), \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$K_2 = A_{n-\nu}^{\kappa-1} \big/ A_n^{\kappa}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $$\Sigma A_n^{\kappa} x^n = (1-x)^{-\kappa-1},$$

$$K_3 = \kappa\omega^{-\kappa}(\omega-u)^{\kappa-1}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

we get $T(t) = \phi_\kappa(t)$, $U_n = s_n^{\kappa}$, $V(\omega) = \tau_\kappa(\omega)$, the Cesàro and Rieszian means of order κ of the function and series. Hardy and Littlewood's theorem then takes the form: *If* $T(t) \to s$ *as* $t \to 0$ *for a given* $\kappa = a$, *then* $U_n \to s$ *as* $n \to \infty$ (*or* $V(\omega) \to s$ *as* $\omega \to \infty$) *for all sufficiently large* κ §. *Conversely, if* $U_n \to s$ *as* $n \to \infty$ (*or* $V(\omega) \to s$ *as* $\omega \to \infty$) *for a given* $\kappa = a$, *then* $T(t) \to s$ *as* $t \to 0$ *for all sufficiently large* κ ‖.

The point I wish to emphasise now is the "mimicry" between the kernels K_1 and K_3—this is also noticeable, but less obvious, in

* A paper read to the British Association, 1933.

† *Math. Zeitschrift*, 19 (1924), 67-96. This theorem has been extended and refined by several writers. For references to relevant papers see a paper by the present writer shortly to appear in the *Proc. London Math. Soc.*

‡ We shall suppose for simplicity that $\phi(t)$ is integrable L and even, that its Fourier series is $\sum_1^\infty a_n \cos nt$ and that $x=0$.

§ Actually for $\kappa > a$.

‖ Actually for $\kappa > a+1$.

the case of K_2. The phenomenon is still present in certain cases if we employ other kernels. Thus, for instance, if we take

$$K_1 = \{\log(1/t)\}^{-1} t^{-\kappa}(t-u)^{\kappa-1}, \quad \ldots\ldots(7)$$
$$K_2 = \{\log \omega\}^{-1} \omega^{-\kappa}(\omega-u)^{\kappa-1},$$

we get an analogue of Hardy and Littlewood's theorem, and it is still true if we replace $\log x$ by a more general function $L(x)$ of sufficiently regular growth. Or again, employing Rieszian means of logarithmic type, we can take

$$K_1 = \kappa\{\log(1/t)\}^{-\kappa}\{\log(u/t)\}^{\kappa-1}, \quad \ldots\ldots(8)$$
$$K_2 = \kappa\{\log \omega\}^{-\kappa}\{\log(\omega/u)\}^{\kappa-1},$$

and the same mimicry is noticeable. That is one kind of relation between function and series at a point, and it is based on the idea of generalised limit and generalised sum.*

After the properties $T(t) \to s$ and $V(\omega) \to s$ it is quite natural to investigate properties like

$$\int_0^\eta |\, T'(t)\,|\, dt < \infty, \quad \ldots\ldots(9)$$

and

$$\int_0^\infty |\, V'(\omega)\,|\, d\omega < \infty, \quad \ldots\ldots(10)$$

or

$$\sum_1^\infty |\, U_n - U_{n-1}\,| < \infty. \quad \ldots\ldots(11)$$

In the last case it is easily seen that U_n tends to a limit as $n \to \infty$, and it is said to do so *absolutely*.

The first results on the absolute summability of Fourier series were given by J. M. Whittaker.† Let

$$V(\omega) = \Sigma a_n e^{-n/\omega} = \frac{1}{\omega}\int_0^\infty e^{-u/\omega} s(u)\, du, \quad \ldots\ldots(12)$$

and suppose (10) holds. Then the series Σa_n is said to be absolutely summable (A), and its sum is $\lim_{\omega\to\infty} V(\omega)$, which necessarily exists—the series is thus also summable (A).

The sufficient condition given by Whittaker for the Fourier series of $\phi(t)$ to be absolutely summable (A) for $t=0$ was

$$\int_0^\eta \frac{|\,\phi(u)\,|}{u}\, du < \infty. \quad \ldots\ldots(13)$$

Later B. N. Prasad ‡ showed that this could be replaced by

$$\int_0^\eta \frac{|\,\phi_1(u)\,|}{u}\, du < \infty, \quad \ldots\ldots(14)$$

* The results above were obtained in collaboration with Dr. A. C. Offord, and some of them will be published in *Compositio Mathematica*.

† *Proc. Edinburgh Math. Soc.* (2), 2 (1930-31), 1-5. The definition of absolute summability (A) given here is equivalent to Whittaker's.

‡ *Ibid.*, 129-134, and *Proc. London Math. Soc.* (2), 35 (1933), 407-424.

or again by

$$\int_0^\eta |d\phi(u)|\, du < \infty . \quad \text{.....(15)}$$

Still more general conditions are

$$\int_0^\eta \frac{|\phi_\alpha(u)|}{u}\, du < \infty , \quad \text{.....(16)}$$

and again

$$\int_0^\eta |\phi_\alpha'(u)|\, du < \infty . \quad \text{.....(17)}$$

The last condition is the most general and is of the form (9), with K_1 given by (4), while the conclusion is of the form (10), with

$$K_3 = \omega^{-1} e^{-u/\omega}.* \quad \text{.....(18)}$$

The next stage of the problem is to try a Cesàro kernel in the series transformation. Taking K_2 as in (5) we ask: *If* (17) *is true for a given* $\kappa = a$, *is* (11) *true*, i.e.

$$\Sigma |s_n^\kappa - s_{n-1}^\kappa| < \infty , \quad \text{.....(19)}$$

for a sufficiently large κ ? † Or again, with K_3 as in (6), *is* (10) *true*, i.e.

$$\int_0^\infty |r_\kappa'(\omega)|\, d\omega < \infty , \quad \text{.....(20)}$$

for a sufficiently large κ ? In both cases the answer is in the affirmative. In fact we have only to take $\kappa > a$, provided also $\kappa > 1$. The last condition is to ensure that the property should depend only on "local" conditions.

Finally, we consider the converse problem. *If either* (19) *or* (20) *holds for a given* $\kappa = a$, *is it true that* (17) *holds for a sufficiently large* κ ? Again we have a favourable answer, and have only to take $\kappa > a + 1$.‡

The combined results may be stated as follows: *A necessary and sufficient condition that the Fourier series should be absolutely summable* (C) *is that* (17) *should hold for some* κ.

Once more we notice the mimicry of kernels, and no doubt there are other results of the same type. The ideas of this problem could probably have been worked out in terms of Fourier integrals. There are also analogous results for allied series, and probably for power series § and Dirichlet series. ‖ It might be interesting to work these out also.

L. S. B.

* These results will be published in the *Proc. Edinburgh Math. Soc.*

† The series would then be said by Fekete to be absolutely summable (C, κ). M. Fekete, *Math. és. Természt Ért.* (1911), 719-726. E. Kogbetliantz, *Bull. des Sci. Math.* (2), 49 (1925), 234-256. A series which is absolutely summable (C) is also absolutely summable (A). M. Fekete, *Proc. Edinburgh Math. Soc.* (2), 3 (1932), 132-134.

‡ So far I have only considered the cases where α is an integer.

§ Compare Hardy and Littlewood, *loc. cit.*, and Bosanquet, *Annals of Math.* (2), 33 (1932), 758-77.

‖ Compare a paper by Offord to be published in the *Proc. London Math. Soc.*

ABELIAN INTEGRALS ATTACHED TO ALGEBRAIC VARIETIES.*

By W. V. D. Hodge.

The purpose of this article is to describe briefly some investigations of the properties of certain integrals associated with the Riemannian manifold of an algebraic variety, which, besides having an interest of their own, appear to be a fruitful source of new properties of the more classical Abelian integrals usually considered in connection with algebraic varieties. The investigations are still at a very early stage, and up to the present only deal with the Abelian integrals of the first kind (those which are without singularities), and in what follows we shall limit our attention to these.

Anyone with an elementary knowledge of the theory of algebraic functions of a complex variable is familiar with the idea of a Riemann surface. The Riemannian manifold of an algebraic variety of m dimensions

$$f(x_0, x_1, \ldots x_m) = 0$$

is generally described as the real locus of $2m$ dimensions which bears to the algebraic variety the same relation as a Riemann surface does to the algebraic curve which it represents. This implies that the points of the manifold are in a unique continuous correspondence with the complex points of the variety. In this sense, the Riemannian manifold has been used by nearly every writer on algebraic functions of m complex variables. The properties of the manifold which have been used are chiefly topological, particular use having been made of the presence of sub-manifolds, corresponding to the algebraic curves, surfaces, etc., which lie on the given variety. The theorems concerning the Abelian integrals associated with the variety have therefore been obtained largely by inductive methods, and recent experience has shown that further progress on these lines is bound to be very slow. Reflection, however, will reveal the fact that we have been taking a somewhat one-sided view of the rôle of a Riemann surface in stressing its topology to the exclusion of other properties. For let us think of the Riemann surface as formed by n complex z-planes branching in some prescribed fashion. In this way we are given a certain metric in which the element of length is $|dz|$. Exception must be made of the critical points, but as we shall see this is really an accident due to the method of constructing the surface. It is the presence of this metric which enables us to develop the theory of analytic functions on the Riemann surface. If we can extend this idea of a metric to a Riemannian manifold, we shall be able to make the manifold a more powerful instrument for dealing with Abelian integrals than it has been hitherto, and this can be done as follows.

It is a known theorem of algebraic geometry that any curve can be transformed birationally into a curve, lying in a space of a suitable number of dimensions, which has no multiple points. A similar

* A paper read to the British Association, 1933.

theorem is true for surfaces. The general theorem has not yet been proved, but we shall assume its truth, or better, we shall confine our attention to varieties which can be transformed into varieties without multiple points, and we shall suppose that V is a variety of m dimensions without singularities, lying in a complex projective space of r dimensions $(z_0, \dots, z_r)$. The Riemannian manifold of this projective space is obtained by writing

$$X_h = \sqrt{2}\,.\,z_h\bar{z}_h,$$
$$X_{hk} = z_h\bar{z}_k + z_k\bar{z}_h = X_{kh} \quad (h, k = 0, \dots r),$$
$$Y_{hk} = i(z_h\bar{z}_k - z_k\bar{z}_h) = -Y_{kh}.$$

z_h, $\bar{z}_h$ are conjugate complex numbers, and we suppose, as is allowable, that the coordinates in our space are restricted by the relation

$$|z_0|^2 + \dots + |z_r|^2 = 1.$$

We now interpret (X_h, X_{hk}, Y_{hk}) as rectangular Cartesian coordinates in a Euclidean space of $(r+1)^2$ dimensions. The locus defined by the above equations can be easily seen to be in (1, 1) correspondence with the complex points of our projective space, and is therefore its Riemannian manifold. The sub-manifold M which corresponds to the points of V is the Riemannian manifold of V. To investigate local properties of V we express its coordinates as functions of m complex parameters

$$x^1 + ix^{m+1}, \dots, x^m + ix^{2m}.$$

$(x^1, \dots x^{2m})$ can then be taken as local parameters on M. M is given as a locus in a Euclidean space, and we are therefore able to speak of a metric on it, given by a quadratic differential form

$$ds^2 = g_{ij}\,dx^i\,dx^j,$$

with the usual summation convention. This metric is of considerable interest in the theory of quadratic differential forms.

When the variety is a curve, $m = 1$, and it can be shown that M is a conformal representation of the n-sheeted Riemann surface of the curve. The Abelian integrals of the first kind on the curve are obtained as complex functions $\Phi + i\Psi$ on M where Φ and Ψ are harmonic functions on M, and

$$\frac{\partial\Phi}{\partial x^1} = \frac{\partial\Psi}{\partial x^2}, \quad \frac{\partial\Phi}{\partial x^2} = -\frac{\partial\Psi}{\partial x^1}$$

which are finite everywhere on M. To find them we have therefore to find the everywhere-finite harmonic functions on M. In order to see how to generalise these integrals we express their properties in the more convenient language of tensor calculus. An everywhere-finite harmonic function on M is not one-valued, but if we write

$$d\Phi = B_1\,dx^1 + B_2\,dx^2,$$

(B_1, B_2) is a one-valued covariant tensor of the first rank. The integrability condition

$$\frac{\partial B_1}{\partial x^2} = \frac{\partial B_2}{\partial x^1}$$

is expressed by saying that the gradient of the tensor is zero, and the harmonic condition

$$\frac{\partial B_1}{\partial x^1}+\frac{\partial B_2}{\partial x^2}=0$$

is expressed by saying that the divergence of the tensor is zero. All these ideas can be extended to covariant tensors of rank higher than the first; we are now in a position to make our generalisations to the Riemannian manifold M of an algebraic variety of m dimensions.

An integral

$$\frac{1}{p!}\int B_{i_1\dots i_p}dx^{i_1}\dots dx^{i_p} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(i)}$$

is said to be harmonic if the anti-symmetric tensor has its gradient zero and its divergence zero. The vanishing of the gradient implies integrability and the vanishing of the divergence implies the integrability of

$$\frac{1}{(n-p)!}\int \Sigma \pm \sqrt{g}\,.\,g^{i_1j_1}\dots g^{i_pj_p}B_{j_1\dots j_p}dx^{i_{p+1}}\dots dx^{i_{2m}}, \quad \dots\dots\text{(ii)}$$

where $(i_1, \dots i_{2m})$ is a derangement of the first $2m$ integers, and the summation is over all derangements, the negative sign being taken with the odd derangements; g is as usual the determinant $|g_{ij}|$, and the numbers g^{ij} are defined by the equations

$$g^{ij}g_{jk}=0, \text{ if } i\neq k$$
$$=1, \text{ if } i=k.$$

The relations between the integrals (i) and (ii) are reciprocal except perhaps for a sign. We shall speak of these as conjugate harmonic integrals. An existence theorem can be established which proves that there are R_p independent p-fold harmonic integrals which are finite everywhere on M, where R_p is the topological invariant known as the pth Betti number.

Various theorems can be established concerning the harmonic p-fold integrals which exist on M. Perhaps the most important to geometers is that which states that the real and imaginary parts of the algebraic p-fold integrals of total differentials of the first kind attached to V are harmonic integrals. On this account it seems reasonable to suppose that by investigating the properties of harmonic integrals, new light may be thrown on the algebraic integrals. This is not the place to enter into details, and indeed it must be confessed that only the most elementary applications have been made up to the present. I shall now, however, mention one or two results which have been obtained by this means.

Let $$\int B_i\,dx^i$$

be a simple harmonic integral which is finite everywhere on M. Then, if

$$C_i=B_{m+i}, \quad C_{m+i}=-B_i,$$

C_i is also a covariant tensor on M, and

$$\int C_i dx^i$$

is also a harmonic integral. Then

$$\int (B_j - iC_j) dx^j$$

is an algebraic (or Picard) integral of the first kind attached to V, and the number of such integrals is $\frac{1}{2}R_1$. This is, of course, not a new result, but the present derivation has certain points of interest.

When we pass to double integrals we begin to get something new. Let us consider, for example, the case $m=2$, $p=2$. The real part of a double integral of the first kind attached to the surface V is a harmonic integral. If it is taken as the integral (i) above, we find that the conjugate integral (ii) is the same integral with the sign changed. Now it can be shown that the harmonic integrals which are related to their conjugates in this way are linear combinations of the real and imaginary parts of the Abelian integrals of the first kind, and one other integral

$$\iint h_{ij} dx^i dx^j,$$

where $h_{ij} = g_{i\,m+j}$, $h_{i\,m+j} = -g_{ij}$.

Thus the total number of such integrals is $2p_g + 1$ where p_g is the geometric genus of the surface.

Now there are certain theorems which can be established concerning the periods of harmonic integrals, and from these it is possible to determine the number of harmonic integrals which are related to their conjugates in the manner described, as a topological invariant of M. Thus p_g can be expressed as a topological invariant. This invariant depends on the matrix giving intersection numbers of the two-dimensional cycles of M. The matrix in question is a symmetric matrix a_{ij}, and can be used to define a quadratic form

$$a_{ij} x^i x^j.$$

If, by a real transformation of the x's, this is transformed to the sum or difference of R_2 squares, the number of negative terms is the same whatever transformation is used, and is hence an invariant of the matrix, and hence of M. This is the invariant in question, which is equal to $2p_g + 1$.

The two examples which I have given are perhaps the simplest which can be given of the applications of our methods to the theory of Abelian integrals. As the multiplicity of the integrals increases, so does the complexity of the problem, but as so little has been done yet in the study of harmonic integrals, it would be rash to prophesy on the success or limitations of the method I have indicated. The whole theory abounds in problems still unsolved, and there is room for much more investigation than it has yet received, and the only safe prophecy which can be made is that the investigator will be repaid for his trouble by the interest which he will derive.

W. V. D. H.

THE TEACHING OF GEOMETRY.

By E. H. Neville.

At a summer school for teachers held at Oxford in 1924 Professor Neville delivered a lecture with the above title. Some explanation of the publication of a ten-year-old manuscript may be desirable. The address deals with a topic of enduring interest, and, in addition, the editor feels that, since members of the Association have been asked to consider this subject in view of the approaching revision of the Report of 1923, the opinions of the Chairman of the Committee which drew up that Report will be particularly valuable at this time.

This article represents the first half of the lecture; the second half, which followed after a few minutes, dealt with congruence, with the theory of parallels, and with proportion and irrationals. The manuscript is incomplete, but enough remains on the first two subjects to form the basis of a second article to be published later.

In the course of centuries many hard things have been said against Euclid, that is, against the use of the *Elements* as a beginners' textbook. The A.I.G.T. might almost be described as a society of professional sayers of such things, and the M.A. has not yet wholly dissipated, shall I say, this part of its inheritance from the parent body. But I propose for a change to ask what there was of value in the use of Euclid that has been lost, or is in most danger of being lost, and what we can do to retrieve the losses and to avert the dangers. We shall not spend an hour and a half answering this question without coming to grips with some of the problems that are most important to the teacher of geometry.

Consider first, that, in England eighty years ago, in France at an earlier date, the teacher had no doubt as to what was happening in the schoolroom. He was teaching Euclid; he could tell beforehand, as a celebrated Cambridge philosopher is said to be able to do, precisely the point in his reasoning that he would have reached on any given date. As for the pupil, he was learning the same Euclid, and acquiring incidentally but inevitably, it was believed, habits of accurate thought that would serve him throughout his life. Every teacher, stupid or clever, knew where he was and what he was up to.

Will anybody dispute that during the past twenty-five years there has been no such universal comfort? There have been teachers who have merely credited an uncanonical nonconformist substitute with the verbal inspiration that it was unfashionable to allow to the decanonised Euclid: who became as word-perfect in Godfrey and Siddons as they had ever been in Gilbert and Sullivan. And there have been of course the Godfreys and the Siddonses themselves, restlessly constructing coherent system to satisfy the needs which they perceived. But between these extremes there has been the majority of teachers, uneasily aware that to them geometry had become formless.

There is another aspect of the formlessness of geometry. The need of supplementing the Euclidean tale has been felt for centuries. This is not surprising. Euclid did not design the *Elements* to tell his pupils everything they would need to know. I recall a theory

that what Euclid had in mind was a treatise on the regular solids, and that the *Elements* were to contain only the geometrical and arithmetical propositions required for this special purpose. Whether this be true or not, it is impossible to suppose that Euclid regarded the property which we call that of the circle of Apollonius, with which he was perfectly familiar, as intrinsically less elementary or less valuable to the geometer than the classification of irrationals which has aroused the enthusiastic admiration of the few European scholars who have ever made themselves acquainted with his 10th book. But whatever additions were to be made, the framework into which they were to be inserted was provided, and from Propositions * A, B, C to the theory of groups of circles there is little in signless, metrical school geometry that did not find a ready point of attachment. To-day, what is book-work in one course is a group of exercises in another, and the very ideas by which one writer organises his material may be entirely ignored by another; in defiance of all physiological possibilities, one writer's backbone is indistinguishable from another man's appendix.

I would say, then, that the most crying need is for some natural morphology of geometry, and let me hasten to stifle any hopes that may be stirring: I have no scheme to offer you. For my part, I do not believe that there is one scheme that is best for all teachers and all pupils. The progressively disposed teacher, if official regulations leave him free to choose for himself, who is unable to form a sound judgment from the standpoint of his individual needs on any course suggested to him, must always be the prey of the enterprising publisher and the slave of the fashionable reform.

It is interesting to find that we cannot discuss practice except after agreeing about principles; we cannot prefer one framework to another until we have faced the abstract question "What is the object of it all?" Professor Baker † has suggested, and we must all agree with him, that in school geometry three aims are distinguishable, and that in any actual course the aim is probably a mixture, of which the ingredients are the teaching of mensuration, a training in logical thinking, and the arousing of interest in geometrical facts and methods. Mensuration is readily dismissed as a necessary subject but a small one. Experience and expert psychology agree that it is not clearness of thought in the abstract which a course of geometry can develop. If the power of logical thinking is possessed, it will be exercised on any material, and judgments on that material will be sound in so far as the material itself is clear-cut and familiar. Teach geometry as many hours a week as you will, and the boy who did not appreciate the difference between a theorem and its converse the first time it came to his notice, though you may drill into him some means of recognising the confusion as a move which is inscrutably forbidden in playing your geometrical game, will grow none the less into a man at the mercy of the first plausible

* Of Simson's edition.

† In an article on the *Report* contributed to the *Gazette*, xii, 73 (1924, May).

swindler who suggests that "You could be trusted to look after my money" is an equivalent proposition to "I can be trusted to look after your money".

If we admit that mensuration is too small a matter and the transformation of mental equipment too large for us, we are driven back on the third reason for our teaching—to give knowledge of and to evoke interest in geometrical facts and methods. After all, this is a natural enough reason, which only a utilitarian world would expect us to supplement.

We come then to the central problem of designing a course in geometry which shall be intrinsically shapely, and at the same time shall embrace all that we feel to be essential. At first glance, our difficulties seem to be enhanced by the fact that the course is for growing boys. Euclid was writing for grown men, who would not have been studying under him at all if their desire for knowledge in the abstract had not been keen and the passion for logical system already dominant. You are teaching boys who have no option save to listen to you, and your course is to begin at a time when their interests are wholly concrete and to assist in their mental development until abstract relations and logical structure have become attractive in themselves. Small wonder that teachers find their problem so difficult that they long to utilise Euclid's solution of his, different though that was in every essential.

Fortunately the difficulty in this respect can be overcome. The Board of Education led the way many years ago in official recognition that the character of geometrical teaching must change as the pupil develops, and the M.A. report divides the course into five stages, of which the first three are characterised as experimental, deductive, and systematising. Stage A, the preliminary experimental stage, must not take too long, and Stage C, in which logic and organisation are to have explicit attention, has not to be tackled by pupils who are reluctant. It is Stage B into which the great mass of school geometry falls; this stage has not to be strained to reach either extreme, and yet the teacher who has a satisfactory programme for Stage B will have no trouble in working up to this stage or away from it.

What, then, are the characteristics of this important stage as conceived by the Committee of the M.A.?

First and foremost I will put the one characteristic that is vital—I dare not say *literally* vital, but at any rate in the strictest metaphorical sense. Nothing is to be "proved" which every healthy youngster would take for granted. What could be better calculated to foster a distrust of words and a contempt for the teacher of geometry than a solemn assurance that a circle has been *proved* to be symmetrical? If we have nothing better to do than that, we have no claim on the fleeting moments of youth that we could maintain for five minutes. It is a tragic cry, most vocal recently in *Back to Methuselah* and familiar to us all, that life is so short that the most willing have no time to learn. Let not our pupils have reason to suspect that we have misused our hold on their scanty hours.

Is it then, some of you will be asking, *not* possible to establish the symmetry of the circle? Surely symmetry is definable. Of course the symmetry of the circle can be related logically to other facts equally obvious. The inter-relations of obvious geometrical facts are not always obvious in themselves, and the formal exposition of these inter-relations is an important part of geometry, but in this process the elementary facts are *organised* rather than proved, and it is the inter-relations which are exposed. This work is not the preposterous word-spinning, the deliberate perversion of instinct, that it is sometimes made to appear, but quite emphatically it does not belong to the early stage in which the misuse of the *Elements* placed it, to the mystification of many generations.

To say that we must not pretend to prove the obvious is not to say that we are not to dwell upon aspects of the obvious, or to indulge in clear statements of the facts as we perceive them. On the contrary, occasions are likely to arise in which one boy needs to be convinced of something that another regards as obvious—for example, that in a figure symmetrical about a line, corresponding lines cannot meet except on the axis.

But as a general tendency, if we are to take the obvious for granted, we must be proving the unexpected or following up the clues of practical geometry to discover whether plausible suggestions are true. Surely this is the secret of vitalising the subject. If my own recollection of emotion is to be trusted, I am a mathematician because parallelograms on the same base and between the same parallels have the same area, and because the angle in the segment of a circle is constant. I cannot remember being excited by Pythagoras' theorem, but I am not surprised to think that I took the rest of Books I and II [illegible] 35, and the first half of Book III quite coolly.

It has been suggested by several writers that among the straightforward propositions with startling dénouements those in which concurrencies and collinearities are involved are the most valuable. These propositions are qualitative rather than quantitative in essence: this is not to say that they necessarily belong to descriptive geometry rather than to metrical, for of course the medians and the perpendiculars need midpoints and right-angles. The propositions also lend themselves to experimental introduction, but here a warning seems wanted. We must not allow boys to suppose that because the triangles measured in a large class give angle-sums between 175° and 185° therefore the angle-sum is the constant 180°, and if the better drawings indicate concurrence, that is not more than a reason for undertaking a theoretical investigation. I must confess that in this matter I do not understand Professor Baker, who seems to expect a standard of draughtsmanship so accurate as to carry complete conviction. I spent hours as a boy trying to make a diagram to exhibit Feuerbach's theorem; I do not remember that I ever got as far as a nine-points circle that really looked like one, and the finer my lines the less credible were any of the theoretical results! I for one could never "get at such things" by making

diagrams, and I am unwilling to believe that my most careful work has always been absurdly below the level of the average boy drawing without prevision. Following another suggestion of Professor Baker's, only a few weeks ago I drew a number of different quadrilaterals to find the fourth harmonic of a given point with respect to two others; the diagram was, I protest, as good as nine out of ten boys would produce, and better than nine out of ten text-book figures, but in the nature of things if one chooses a quadrilateral somewhat exaggerated in form, the guidance to the final point is uncertain; all one can really be entitled to say on the evidence is that for ordinary forms of the quadrilateral the variation in position of the fourth point seems to be slight, as if a stationary position was in question. Of course I agree with Professor Baker that a geometry in which the uniqueness of the fourth harmonic as given by a quadrilateral construction is a fundamental axiom is fascinating in its generality, but I am not convinced that schoolboys can dispense with ordinary axioms of congruence, nor am I sure that Professor Baker means that they should, since he supposes them already to have found experimentally that the midpoints of the diagonals of a complete quadrilateral are collinear.

We have come then to this understanding, that in the main course in geometry, which is all that we are concerned with at present, we will accept everything which is obvious, not analysing the interrelations of the assumptions, but not being afraid on occasion to state an assumption in explicit terms. We will learn to appreciate the obvious from as many points of view as possible, but our design shall be to watch the emergence of the unexpected results and to verify guesses made on empirical grounds.

There will be little dispute about the [illegible] of the course, nor will anyone deny that there is to be a complete break from the Greek style in the recognition of generalisations. We shall not separate the rectangle property of the circle into two cases, one for an external point, the other for an internal, and we shall represent Pythagoras' theorem and its extensions as the cases of a single formula. It follows that we shall introduce very early the notion of attaching sign to geometrical measurements, and that we shall not be afraid of using the trigonometrical ratios. Our problem still remains substantial enough: we wish to frame a course that will cover the ground, I was going to say, but as that phrase seems redolent of plane geometry, I had better say that will explore space both adequately and systematically in a search for facts that are not obvious, and we must not forget the requirement with which we began, that our course is to be as far as possible a unified structure.

I suggest at this point, with diffidence and to provoke discussion, that in school geometry system is not to be based on unity of method. Unity of method is perhaps the only charm which can render an elementary text-book endurable, not to say attractive, to the mature mind, but the learner should be made acquainted during his school life with a large variety of tools and should be exercised in the rejection of inappropriate methods.

As Whitehead said long ago, every method creates its own field of activity. No impression can be more disastrous to give, whether as part of a geometrical education or as part of the training of a mathematician, than that to be master of one method is to hold the key to all geometry. This is most strikingly true when we consider the limitations of Cartesian geometry—has anyone ever proved the theorem of the double-six by writing down the equations of all the lines involved? But the principle is sound even within the range of pure geometry; symmetry, rotation, similarity, mean centres, no one order of ideas must be allowed to carry all the weight even if an enthusiastic advocate is prepared to show that it would not collapse under the strain.

There is one generalisation which seems wide enough to embrace most of the ideas essential in elementary geometry, and which certainly gives the best outlook towards more advanced work. Can we lay more stress on correspondences between component elements of different figures, or of different parts of the same figure? Reflection, rotation, magnification all come under this head as simple types of correspondence, and if the emphasis is on the correspondence, the homogeneity in these simple examples will not seem unduly important and we shall be prepared presently to find circles in one figure corresponding to lines in another, or points in one to planes in another. This is just one of those plausible suggestions that are almost worthless until they are put to the test. You cannot estimate a scheme except by working it out in minute detail. You must write down every proof in order to be sure that you are not dealing illogically with the steps that involve theorems which are not obvious, and you will not know whether everything that you think important has found a natural place in your scheme until you have constructed and reconstructed with tireless patience.

A last word on this subject. Why, after all, are we aware that we have not reached finality in a matter which has been discussed for two thousand years? I will suggest that it has only lately become possible for any mathematician to see truly the difference between proving that the angles at the base of an isosceles triangle are equal, and proving that the angle in a semicircle is a right-angle; the explicit recognition of the distinction between deduction and organisation is a recent achievement, and we need feel no surprise that it must exercise over school teaching a profound influence. When this influence has permeated, will there be a period of security, in which there will be so little room for difference of opinion that the price paid for conformity to some one canon would not be excessive? What differences of opinion will be debated in this room twenty years hence I do not know. But I know that though a tinge of opprobrium will cling always to the names of nonconformist, reformer, heretic, revolutionary, the society in which men are not bred who earn these names is a doomed society, and I believe that the profession to which you belong will never become one in which every soul that bears the pains of travail must suffer also the chill of discouragement and isolation.

E. H. N.

PROBLEMS CONNECTED WITH A REGULAR POLYGON OF N SIDES.

By J. Travers.

In connection with a regular polygon of N sides, many neat and interesting problems can be set, and it is the object of the present article to focus attention on these in order that the complete solution of all these problems in the case of an even-sided polygon when N is a multiple of 3 may be completely determined.

The easiest problem is to find the number of *diagonals*. This number is evidently given by $N(N-3)/2$, and is the same for all regular polygons whether N is odd or even.

In most of the problems that follow we must distinguish between the cases in which the number of sides is odd or even, and hence we will deal with an odd-sided regular polygon first.

Assuming N is odd we get the following results.

To find the number of pericyclic *quadrilaterals* into which an odd-sided polygon is divided when all its diagonals are drawn.

Clearly the number is given by $N(N-1)(N-2)(N-3)/24$, or calling the number of pericyclic quadrilaterals Q, and denoting the number of diagonals by D, we find $Q=D(D+1)/6$. As each of these quadrilaterals has two diagonals which have one point of *intersection*, it is clear that the number of *intersections* is the same as the number of quadrilaterals. Our next problem is to find the number of *compartments* into which a polygon is divided when all its diagonals are drawn. Before drawing any diagonal we have one compartment. As successive diagonals are drawn, each diagonal or its intersection with another gives us an additional compartment. Hence the total number of *compartments* is given by $1+D+Q$, or in terms of the diagonals we have number of *compartments* is given by $(D+1)(D+6)/6$. Our next problem is to find the number of *bits* into which the diagonals of a polygon divide each other. Since each intersection gives us two bits, and each diagonal gives us one bit, the total number of *bits* is given by $2Q+D$, or in terms of the diagonals we get number of *bits* $=D(D+4)/3$.

Our next task is to find the number of triangles that are formed when all the diagonals of a polygon are drawn. For this purpose we will divide the triangles into four classes.

Class A. Those having three vertices on the perimeter of the polygon. These are evidently given by $N(N-1)(N-2)/6$.

Class B. Triangles having two vertices on the perimeter of the polygon. Since each intersection gives us four triangles of this class we see that the number of triangles having two vertices on the perimeter of the polygon is given by $N(N-1)(N-2)(N-3)/6$, or in terms of the diagonals $2D(D+1)/3$.

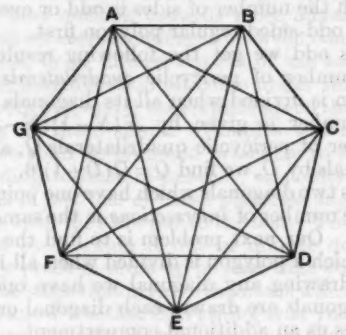
Class C. Triangles having one vertex on the perimeter of the polygon. By drawing a figure we can readily determine the law that governs the formation of triangles of this class.

Take the vertex A in the figure $ABCDEFG$.

GB meets diagonals through A in 4 places, forming 6 triangles.
GC meets diagonals through A in 3 places, forming 3 triangles.
GD meets diagonals through A in 2 places, forming 1 triangle.
FB meets diagonals through A in 3 places, forming 3 triangles.
FC meets diagonals through A in 2 places, forming 1 triangle.
EB meets diagonals through A in 2 places, forming 1 triangle.

Hence the total triangles formed at the vertex A is given by $1+4+10\ldots$ to $(N-4)$ terms. But this is the sum of the first N figurate numbers to $(N-4)$ terms, and is given by

$$(N-4)(N-3)(N-2)(N-1)/24.$$



But there are N vertices, and hence the total number of triangles of class C is given by $N(N-1)(N-2)(N-3)(N-4)/24$, or in terms of the diagonals by $D(D+1)(N-4)/6$.

CLASS D. Triangles having no vertex on the perimeter of the polygon. In order to get a triangle of this class we must select six points on the perimeter of the polygon, and we can select six points out of N points in $N(N-1)(N-2)(N-3)(N-4)(N-5)/720$ ways, or in terms of the diagonals $D(D+1)(D+10-3N)/90$.

The triangles mentioned under classes A, B, and C may be joined into one class under the heading of *triangles having at least one vertex on the perimeter of the polygon*, and this number is given by $N(D+1)(D+2)/6$.

The formulae for classes A, B, and C are the same in the case of an even-sided polygon, but class D is not the same, and we must now derive formulae to deal with the total triangles formed when the polygon has an even number of sides. Taking the number of triangles having at least one vertex on the perimeter of the polygon as correct for both odd- and even-sided polygons, we have only to deal with the number of triangles of class D that *vanish* owing to the

concurrencies of three or more diagonals. For this purpose we must divide the concurrencies into *three classes*.

1. Central concurrencies, giving rise to $R(R-1)(R-2)/6$ vanishing triangles, where $R=N/2$, and is the same in all even-sided polygons.

2. Diametrical concurrencies, occurring on the diagonals that pass through the centre, giving rise to $R(R-2)(R-2)/2$ where R is even, and to $R(R-1)(R-3)/2$ where R is odd. The proof for these formulae is easily derived from the consideration of the number of concurrencies on a particular diameter. This number is given by

$$(R-2+R-3+R-4 \ldots + \ldots 1)-(R-2)/2=(R-2)(R-2)/2$$

when R is even, and by

$$(R-2+R-3+R-4 \ldots + \ldots 1)-(R-1)/2=(R-1)(R-3)/2$$

when R is odd.

3. Concurrencies occurring on the bisectors of the angles between successive diameters, and giving rise to $R(R^2-6R+8)/3$ triangles when R is even, and to $R(R^2-6R+5)/3$ when R is odd.

By adding the results of the above concurrencies we get the total vanishing triangles :

(1) When R is even $=R(R-2)(2R-5)/2$, and

(2) When R is odd $=R(R-1)(2R-7)/2$.

Now subtract the number of vanishing triangles from the number in class D, and we get the number of triangles that have no vertex on the perimeter of the polygon, when N is even.

There is a fourth class of vanishing triangle that occurs in an even-sided polygon when N is a multiple of 3, and greater than 6. These vanishing triangles arise when there are concurrencies of three or more lines that occur, not on the bisectors of the angles between successive diameters, but the total triangles formed by these is not yet known.

J. TRAVERS.

---

947. Letter from Attorney-General to Master of Jesus College, Cambridge, 1800, *re* a scholar at Sevenoaks.

"His father . . . is told that considerable stress being laid [at Jesus] upon Mathematical Science, for which in general boys at Sevenoaks are not sufficiently put upon train, it will be necessary for him to be a hard student 8 or 9 hours a day for the purpose of acquiring ye knowledge . . . that will ensure to him the object I have stated." [The object was a fellowship.]

Letter from the student, to his father, a little later.

"I am labouring hard to keep up with the young men who come out of the North in Mathematicks ; they come up having all Euclid and the greater part of Algebra at their fingers' ends, so that it is with great difficulty that I can keep with them. They have left all from the South behind them except me, and I hope that with great assiduity I shall be able to do for them."—*Sevenoaks School*, 1432-1932, p. 73. [Per Mr. A. P. Rollett.]

NOTE ON THE NATURE OF TRIGONOMETRY.

By V. Naylor.

Trigonometry, like most abstract sciences, has both an operational side and a quantitative side.

The operational side is dominated by the hypothesis that a function $f(x)$ exists which has the property

$$(1) \qquad f(x)f(y)=f(x+y)\,;\ f(0)=1.$$

It is from this that the usual rules for operating with the trigonometric functions are deduced.

If $f(x)$ is such a function, we may define two functions $s(x)$, $c(x)$ by

$$(2) \qquad s(x)=\frac{f(x)-f(-x)}{2i},\quad c(x)=\frac{f(x)+f(-x)}{2}.$$

From these definitions we see at once that

$$(3) \qquad s(0)=0,\ c(0)=1,\ s(-x)=-s(x),\ c(-x)=c(x).$$

Also,

$$(4) \qquad \begin{aligned} c(x)+i\,.\,s(x)&=f(x),\\ c(x)-i\,.\,s(x)&=f(-x),\end{aligned}$$

and hence

$$c^2(x)+s^2(x)=1.$$

From (1) and (4), we have

$$f(x+y)=f(x)f(y),$$

$$c(x+y)+i\,.\,s(x+y)=[c(x)+i\,.\,s(x)][c(y)+i\,.\,s(y)],$$

and hence

$$(5) \qquad \begin{aligned} s(x+y)&=s(x)c(y)+c(x)s(y),\\ c(x+y)&=c(x)c(y)-s(x)s(y).\end{aligned}$$

De Moivre's theorem holds; for, from (4) and (1),

$$(6) \qquad \begin{aligned}[c(x)+i\,.\,s(x)]^n=[f(x)]^n&=f(nx)\\ &=c(nx)+i\,.\,s(nx).\end{aligned}$$

Although the usual trigonometric identities hold for $c(x)$ and $s(x)$, yet $c(x)$ is not $\cos x$ and $s(x)$ is not $\sin x$.

This is seen by considering particular types of $f(x)$:

(7) *If* $f(x)=e^x$, we have

$$s_1(x)=\frac{e^x-e^{-x}}{2i},\quad c_1(x)=\frac{e^x+e^{-x}}{2},$$

which we recognise as being synonymous with

$$\frac{1}{i}\sinh x \text{ and } \cosh x.$$

For this case, $s_1(x)$ and $c_1(x)$ are not periodic, while (4) becomes

$$\left(\frac{e^x-e^{-x}}{2i}\right)^2+\left(\frac{e^x+e^{-x}}{2}\right)^2=1,$$

an identity which is far removed from Pythagoras' theorem.

(8) *If* $f(x)=e^{ix}$, we have

$$s_2(x)=\frac{e^{ix}-e^{-ix}}{2i},\quad c_2(x)=\frac{e^{ix}+e^{-ix}}{2},$$

which we recognise as the usual $\sin x$ and $\cos x$ (with base e).

(9) *If* $f(x)=a^{ix}$, we have

$$s_3(x)=\frac{a^{ix}-a^{-ix}}{2i},\quad c_3(x)=\frac{a^{ix}+a^{-ix}}{2},$$

which we may call "$\sin x$ with base a" and "$\cos x$ with base a". The period of $s_3(x)$ is $2\pi/\log_e a$.

(10) The results of differentiating $s(x)$ and $c(x)$ depend on a peculiar property of the function $f(x)$, defined at (1).

We have

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f(x)\,.\lim\frac{f(h)-f(0)}{h}=f(x)\,.\,f'(0),$$

so that we may replace the operator d/dx by the factor $f'(0)$.

We then have

$$(11)\quad s'(x)=\frac{f'(x)+f'(-x)}{2i}=\frac{f'(0)}{i}\cdot\frac{f(x)+f(-x)}{2}=\frac{f'(0)}{i}\cdot\cos x.$$

This gives, when applied to (7), (8), (9), the results

$$(12)\quad s_1'(x)=\frac{1}{i}\cdot c_1(x);\ s_2'(x)=c_2(x);\ s_3'(x)=\log_e a\,.\,c_3(x).$$

The result (11) is, when $f(x)$ is differentiable, often expressed thus:

$$\lim_{x\to 0}\frac{s(x)}{x}=\lim\frac{f(x)-f(-x)}{2ix}=\frac{1}{i}\cdot f'(0).$$

In the above scheme no use has been made of the concept of radian. We may, however, as with the results of all abstract reasoning, seek to give physical interpretations to the results of our working. If in example (8), for instance, we establish correspondence between the abstract number x and a geometric angle, we get the geometric aspect of ordinary trigonometry.

But there is no obvious geometric interpretation for (7) and (9). All of this goes to show that there are many trigonometries, just as there are many geometries and many algebras. In fact, nowadays, we may use

$$\sin^2 A+\cos^2 A=1,$$

where A is a matrix!

It is interesting to note, if $s_4(x)$ and $c_4(x)$ denote the partial sums of the expansions of $\sin x$ and $\cos x$ when all powers of x after, say, the 5th are neglected, that the usual trigonometric formulae hold for $s_4(x)$ and $c_4(x)$.

Thus we have

$$\begin{aligned} &s_4(x) \,.\, c_4(y) + c_4(x) \,.\, s_4(y) \\ &\quad = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(1 - \frac{y^2}{2!} + \frac{y^4}{4!}\right) + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\right)\left(y - \frac{y^3}{3!} + \frac{y^5}{5!}\right) \\ &\quad = (x+y) - \frac{(x+y)^3}{3!} + \frac{(x+y)^5}{5!} \\ &\quad = s_4(x+y). \end{aligned}$$

To see that the other formulae are true we observe that, if J is an operator which has the property

$$J^m \,.\, J^n = J^{m+n}; \quad J^{m+n} = \begin{cases} 0 \\ 1 \end{cases}, \quad \begin{matrix} m+n \geqq 6 \\ m+n < 6 \end{matrix},$$

then $$e^{Jx} \,.\, e^{Jy} = e^{J(x+y)}.$$

Hence $$s_4(x) = \frac{e^{iJx} - e^{-iJx}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

$$c_4(x) = \frac{e^{iJx} + e^{-iJx}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

and $$c_4(x) + i s_4(x) = e^{iJx}.$$ V. N.

---

948. He [Domine Dobiensis] had proved himself a great mathematician, having, it is said, squared the circle by algebraical false quantities, but would never show the operation for fear of losing the honour by treachery. He had also discovered as many errors in the demonstrations of Euclid as ever did Joey Hume in the army and navy estimates, and with as much benefit to the country at large.—Captain Marryat, *Jacob Faithful*, ch. iii.

"Joey Hume" may be Joseph Hume (1777-1855), one of the leading radicals of his time. He entered Parliament in 1812 and for many years led the radical party; he frequently advocated the abolition of military and naval sinecures. O'Connell said of him that he would have been an excellent speaker, if only he would finish a sentence before beginning the next but one after it. In that section of *A Budget of Paradoxes* which deals with a debate in the House of Commons on a proposed decimal coinage De Morgan says, "Davies Gilbert used to maintain that during the long period he sat in the House, he never never knew more than three men in it, at one time, who had a tolerable notion of fractions. [I heard him give the names of three at the time when he spoke: they were Warburton, Pollock, and Hume. He himself was then out of Parliament.] Joseph Hume affirmed that he had never met with more than ten members who were arithmeticians."

949. The one apparent disadvantage of the invention is that it cannot at present be used as an amplifier; its "magnification factor" is zero. It can, however, be adapted to automatic volume control.—From a paragraph "New Wireless Marvel", *News-Chronicle*, 13th February, 1933. [Per Mr. F. C. Boon.]

MATHEMATICAL NOTES.

1088. *A familiar equation.*

If $\alpha = \pi/7$, the six different values of $\tan k\alpha$ are the roots of the equation

$$x^6 - 21x^4 + 35x^2 - 7 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

Should we notice that this equation can be written in the form

$$(x^3 - 7x)^2 = 7(x^2 + 1)^2,$$

we can infer at once that three of the different values are the roots of the cubic equation

$$x^3 + \sqrt{7}x^2 - 7x + \sqrt{7} = 0$$

and the other three are the roots of the complementary equation

$$x^3 - \sqrt{7}x^2 - 7x - \sqrt{7} = 0,$$

and it is then not difficult to assign the roots correctly. But the transformation of the original equation is by no means obvious, and the separation of the roots is at first sight arbitrary. Is the whole process a curiosity ?

Return to the equation A. If $\tan\theta$ is one root of the equation, then $\tan 2\theta$ is another. In other words, if t is one root, then a second root u can be evaluated as $2t/(1-t^2)$, a third root v as $2u/(1-u^2)$, and so on. It follows that if in A we substitute $2x/(1-x^2)$ for x, we shall obtain a second equation satisfied by $\tan k\alpha$; since, however, this second equation is satisfied by all the roots of A, this substitution cannot help us to resolve A. But there is another line of argument. We have spoken of a second root u, a third root v, *and so on*. But of course we are not going to get an indefinite number of new roots. In fact, if $\tan 7\theta = 0$, then $\tan 8\theta = \tan\theta$, and so from the third root v we shall return to t. It is no use expecting to utilise this result too crudely, and to find in the equation $t = 2v/(1-v^2)$ a means of resolving A, for every root of A satisfies the condition by which the new equation has been determined. If t is replaced by x, the polynomial which is equated to zero in the new equation must contain as a factor the polynomial on the left of A, and it is easy to verify that the new polynomial is of the ninth degree and is obtained by multiplying the original polynomial by $x(x^2+1)$. There is another way of utilising the cycle t, u, v. This cycle consists of only half the roots of A, and the other three roots form a distinct cycle $-t, -u, -v$. If then we can find a property which belongs to one cycle but not to the other, we can obtain an equation B which, though it may have roots which do not belong to A at all, cannot have the roots $-t, -u, -v$, and the equation whose roots are the roots common to A and B will be the equation whose roots are t, u, v.

A suitable property is evident. The product of the six roots of A is -7, and can be equated to $-t^2u^2v^2$. The product

$$\tan\alpha \tan 2\alpha \tan 4\alpha$$

is negative, since α and 2α are acute and 4α is obtuse. We shall therefore obtain the equation whose roots are $\tan\alpha$, $\tan 2\alpha$, $\tan 4\alpha$ if we associate with A the condition $tuv = -\sqrt{7}$. Substituting for u, v in terms of t and replacing t by x, we have

$$\sqrt{7}(x^4 - 6x^2 + 1) + 8x^3 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

We might determine the equation whose roots are t, u, v by finding the highest common factor of the polynomials in A and B. Actually this is unnecessary, for since the product of the four roots of B is 1, the irrelevant root is $-1/\sqrt{7}$, and we can verify that $\sqrt{7}x + 1$ is a factor of the polynomial in B; simple division gives the equation

$$x^3 + \sqrt{7}x^2 - 7x + \sqrt{7} = 0,$$

which is therefore the equation whose roots are

$$\tan\alpha, \quad \tan 2\alpha, \quad -\tan 3\alpha.$$

Even for an isolated problem an intelligent solution is more satisfying than a lucky guess. It will be evident, further, that from the point of view adopted, the equation A is by no means isolated, and the enterprising student will readily construct examples involving other angles of the form $\pi/(n^k - 1)$. Actually the case for considering the equation as I have done goes much deeper. Essentially the equation has been broken into two simpler equations, because the roots have been divided into two groups by an operation which passes from one root to another, but does not pass, however many times repeated, from one root to *any* other, and it is the recognition of the group-operation which leads to the analysis of the equation: we have had a glimpse of the really fundamental principles of the theory of equations. E. H. N.

1089. *The Vector Triple Product.*

The formula for the expansion of the vector product $\mathcal{V}(\mathcal{V}\mathbf{ab})\mathbf{c}$ is the only formula in vector algebra which is in the least degree tiresome to find. To classify writers by their method of proving this result would be interesting, but my present purpose is the more practical one of suggesting a proof which altogether avoids decomposition.

Being perpendicular to $\mathcal{V}\mathbf{ab}$, the vector $\mathcal{V}(\mathcal{V}\mathbf{ab})\mathbf{c}$, which we will denote for brevity by $\mathbf{p}$, is coplanar with $\mathbf{a}$, $\mathbf{b}$, and is therefore of the form $x\mathbf{a} + y\mathbf{b}$, where x, y are scalar coefficients each of which may depend, as far as we yet know, on the three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Hence the projected (scalar) product $\mathcal{G}\mathbf{pd}$, where $\mathbf{d}$ is any fourth vector, is $x\mathcal{G}\mathbf{ad} + y\mathcal{G}\mathbf{bd}$. But if $\mathbf{v}$ denotes $\mathcal{V}\mathbf{ab}$, the product $\mathcal{G}\mathbf{pd}$ is the scalar triple product $\mathcal{G}\{(\mathcal{V}\mathbf{vc})\mathbf{d}\}$, and by the cyclic symmetry of this product, $\mathcal{G}\mathbf{pd}$ can be expressed as $\mathcal{G}\{\mathbf{v}(\mathcal{V}\mathbf{cd})\}$, that is, as

$$\{\mathcal{G}(\mathcal{V}\mathbf{ab})(\mathcal{V}\mathbf{cd})\}.$$

It follows that the expression $x\mathcal{G}\mathbf{ad} + y\mathcal{G}\mathbf{bd}$ depends on $\mathbf{c}$ in the same way as on $\mathbf{d}$, that is, is linear in $\mathcal{G}\mathbf{ac}$ and $\mathcal{G}\mathbf{bc}$. But, further, since the product $\mathcal{G}\{(\mathcal{V}\mathbf{ab})(\mathcal{V}\mathbf{cd})\}$ depends on $\mathbf{a}$ and $\mathbf{b}$ in the same way as on $\mathbf{c}$ and $\mathbf{d}$, the expression is linear in $\mathcal{G}\mathbf{ac}$, $\mathcal{G}\mathbf{ad}$, and

linear also in $\mathcal{G}\mathbf{bc}$ and $\mathcal{G}\mathbf{bd}$. Thus x is linear in $\mathcal{G}\mathbf{ac}$ and $\mathcal{G}\mathbf{bc}$, and since the product $\mathcal{G}\mathbf{ac}\,\mathcal{G}\mathbf{ad}$ cannot occur, x is a numerical multiple of $\mathcal{G}\mathbf{bc}$; similarly, y is a numerical multiple of $\mathcal{G}\mathbf{ac}$. Hence for any four vectors

$$\mathcal{G}\{(\mathcal{V}\mathbf{ab})(\mathcal{V}\mathbf{cd})\} = h\mathcal{G}\mathbf{ad}\mathcal{G}\mathbf{bc} + k\mathcal{G}\mathbf{bd}\,\mathcal{G}\mathbf{ac},$$

where h, k are numerical multipliers independent of all the vectors. Since interchange of c and d alters the sign of the product, $h = -k$. Lastly, if c, d coincide with a, b, we have on the left-hand side $(ab\sin\theta)^2$ and on the right $k\{a^2b^2 - (ab\cos\theta)^2\}$, where θ is an angle between a and b and a, b are amounts of these vectors; hence $k = 1$.

This establishes the formula

$$\mathcal{G}\{(\mathcal{V}\mathbf{ab})(\mathcal{V}\mathbf{cd})\} = \begin{vmatrix} \mathcal{G}\mathbf{ac} & \mathcal{G}\mathbf{ad} \\ \mathcal{G}\mathbf{bc} & \mathcal{G}\mathbf{bd} \end{vmatrix},$$

from which we can return at once to the less symmetrical result

$$\mathcal{V}(\mathcal{V}\mathbf{ab})\mathbf{c} = (\mathcal{G}\mathbf{ac})\mathbf{b} - (\mathcal{G}\mathbf{bc})\mathbf{a}.$$

E. H. N.

1090. *The Central Point and Parameter of the Generator of a Ruled Surface.*

For the determination of the central point and the parameter of distribution of an individual generator of a ruled surface, the usual method, of introducing a neighbouring generator, involves considerations unnecessarily complicated. Let the point Q trace a guiding curve on the surface, and let g be a vector in the generator through QE, a vector, in Hamilton's phrase, emanating from the current point of the curve. The position of Q and the vector g depend on a variable u, and if O is any point on the generator through Q, the step QO is a multiple $v\mathbf{g}$ of the vector g. If v varies while u remains constant, O describes the generator, and the distance QO is proportional to v, being gv, where g is the measure of the vector g.

One vector in the tangent plane at O is the vector g which lies in the generator. A second tangential vector at O is the velocity of O in the curve which O describes if u varies and v remains constant; this vector is $\mathbf{w} + v\,d\mathbf{g}/du$, where w is the velocity of Q in the guiding curve. Hence one vector n in the normal to the ruled surface at O is the vector product, given by

$$\mathbf{n} = \mathcal{V}\mathbf{g}(\mathbf{w} + v\,d\mathbf{g}/du),$$

and if C is any other point of the generator,

$$\mathbf{n}_c = \mathcal{V}\mathbf{g}(\mathbf{w} + v_c\,d\mathbf{g}/du),$$

whence

$$\mathbf{n} = \mathbf{n}_c + (v - v_c)\,\mathcal{V}\mathbf{g}\,d\mathbf{g}/du.$$

This formula expresses a normal vector at O in terms of the known vectors $\mathbf{n}_c$, $\mathcal{V}\mathbf{g}\,d\mathbf{g}/du$, and if these two vectors are perpendicular, we have a decomposition of n into perpendicular components, implying at once that if γ is an angle between the normal at C and the normal at O,

$$\tan\gamma = (v - v_c)b/a = rb/ag,$$

where a measures $\mathbf{n}_c$, b measures $\mathcal{V}g\,dg/du$, and r is the length OC. The condition for $\mathbf{n}_c$ to be perpendicular to $\mathcal{V}g\,dg/du$ is linear in v_c and determines the position of C. The measure of $\mathbf{n}_c$ follows, while the measure of $\mathcal{V}g\,dg/du$ is calculable without any reference to the guiding curve.

Consider, for example, the surface generated by the line

$$x+yu=3u(1+u^2),\quad y+2zu=u^2(3+4u^2)$$

(Bell, 2nd ed., p. 322, Ex. 10). We may take for guiding point the point for which $y=0$, that is, the point $\{3u(1+u^2),\ 0,\ \frac{1}{2}u(3+4u^2)\}$, and for emanant the vector $(2u^2,\ -2u,\ 1)$; the square of the measure of this emanant is $(1+2u^2)^2$, and by taking the measure as $1+2u^2$ we adopt a definite direction of measurement along the generator. For any value of v, the vector $(2u^2v,\ -2uv,\ v)$ is the vector of a step of length $(1+2u^2)v$ along the generator. The coordinates $\{3u(1+u^2)+2u^2v,\ -2uv,\ \frac{1}{2}u(3+4u^2)+v\}$ are those of an arbitrary point O of the surface. One tangential vector at O is the emanant $(2u^2,\ -2u,\ 1)$, another is the vector

$$\{3(1+3u^2)+4uv,\ -2v,\ \tfrac{3}{2}(1+4u^2)\},$$

and one normal vector $\mathbf{n}$ at O is the vector product

$$\{-3u(1+4u^2)+2v,\ 3(1+2u^2-4u^4)+4uv,\ 6u(1+3u^2)+4u^2v\},$$

that is,

$$3\{-u(1+4u^2),\ (1+2u^2-4u^4),\ 2u(1+3u^2)\}+2v(1,\ 2u,\ 2u^2).$$

The normal vector $\mathbf{n}_c$ at a point C is

$$3\{-u(1+4u^2),\ (1+2u^2-4u^4),\ 2u(1+3u^2)\}+2v_c(1,\ 2u,\ 2u^2),$$

and we have

$$\mathbf{n}=\mathbf{n}_c+2(v-v_c)(1,\ 2u,\ 2u^2).$$

This last formula expresses a decomposition of $\mathbf{n}$ into perpendicular components if v_c is given by the equation

$$3\{-u(1+4u^2)+2u(1+2u^2-4u^4)+4u^3(1+3u^2)\}$$
$$+2v_c(1+4u^2+4u^4)=0,$$

that is, by $2v_c+3u=0$. Thus the central point is $(3u,\ 3u^2,\ 2u^3)$. The vector $\mathbf{n}_c$ is $3\{-2u(1+2u^2),\ (1-4u^4),\ 2u(1+2u^2)\}$, that is, $3(1+2u^2)\{-2u,\ 1-2u^2,\ 2u\}$, of which one measure is $3(1+2u^2)^2$, and we can write

$$\mathbf{n}=3(1+2u^2)^2\mathbf{a}+2(1+2u^2)(v-v_c)\mathbf{b}=3(1+2u^2)^2\mathbf{a}+2r\mathbf{b},$$

where $\mathbf{a}$, $\mathbf{b}$ are unit vectors at right angles to each other, and r is the distance of O from C. An angle γ from $\mathbf{a}$ to $\mathbf{n}$ is given by

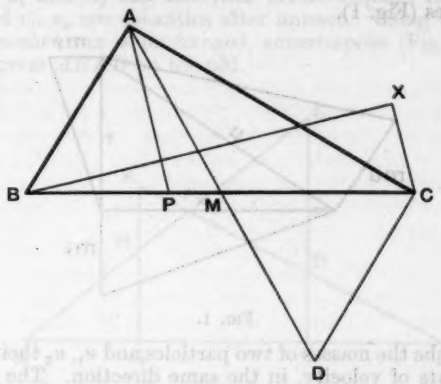
$$\tan\gamma=r/\tfrac{3}{2}(1+2u^2)^2,$$

and this expression for $\tan\gamma$ exhibits one fundamental property of the central point and gives the value of the parameter of distribution.

E. H. N.

1091. *Geometrical proofs of some trigonometrical formulae.*

The following proof of two important trigonometrical formulae is so simple that it is very strange that it has not (to my knowledge) appeared in any of the text-books.



In the triangle ABC, AP is the bisector of the angle A, BX is at right angles to AP. Then $\angle CBX = \frac{1}{2}(B-C)$. Projecting the sides of the triangle ABC on AP,

$$a \sin \tfrac{1}{2}(B-C) = (b-c)\cos \tfrac{1}{2}A\ ;$$

projecting at right angles to AP,

$$a \cos \tfrac{1}{2}(B-C) = (b+c)\sin \tfrac{1}{2}A.$$

Hence
$$\tan \tfrac{1}{2}(B-C) = \frac{b-c}{b+c}\cot \tfrac{1}{2}A.$$

Let the median $AM(m_1)$ make $\angle BAM = \beta$, $\angle CAM = \gamma$. Then

$$\angle PAM = \tfrac{1}{2}(\beta-\gamma).$$

Produce AM to D so that $MD = AM$. Projecting the sides of the triangle ADC on AP,

$$2m_1 \cos \tfrac{1}{2}(\beta-\gamma) = (b+c)\cos \tfrac{1}{2}A\ ;$$

projecting at right angles to AP,

$$2m_1 \sin \tfrac{1}{2}(\beta-\gamma) = (b-c)\sin \tfrac{1}{2}A.$$

Hence
$$\tan \tfrac{1}{2}(\beta-\gamma) = \frac{b-c}{b+c}\tan \tfrac{1}{2}A.$$

J. W. Rogers.

1092. *Graphical representation of Kinetic Energy.*

The kinetic energy of a particle of mass m moving with velocity u may be represented by the area of a right-angled triangle of sides mu and u. For varying values of u the energy triangles of a given

particle will be similar and if u has components $\dot{x}$ and $\dot{y}$ in two perpendicular directions, the relation

$$\tfrac{1}{2}mu^2=\tfrac{1}{2}m\dot{x}^2+\tfrac{1}{2}m\dot{y}^2$$

corresponds to an obvious case of a proposition on the areas of similar figures (Fig. 1).

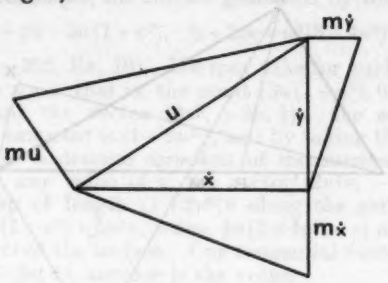


FIG. 1.

Let m_1, m_2 be the masses of two particles and u_1, u_2 their velocities, or components of velocity, in the same direction. The sum of the kinetic energies is represented by the area $ABDC$ (Fig. 2).

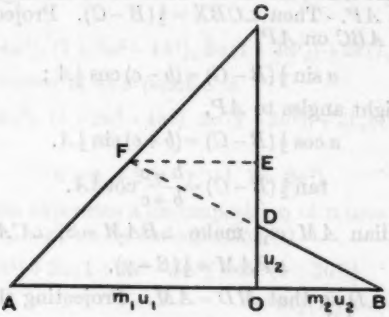


FIG. 2.

Let BD produced meet AC in F; draw FE parallel to AB. Then

$$\begin{aligned} OE &= \text{height of } \triangle AFB \\ &= \frac{AB}{\cot A+\cot B} \\ &= \frac{m_1u_1+m_2u_2}{m_1+m_2} \\ &= \text{velocity of the centre of mass.} \end{aligned}$$

Thus CE and DE represent the velocities relative to the centre of mass.

Total area $=\triangle AFB+\triangle CFE+\triangle DFE$,

that is, total K.E. = K.E. of both particles moving with the centre of mass + K.E. of their motion relative to centre of mass.

If u_1 and u_2 have opposite directions we may reverse both OB and OD and use the same construction.

Suppose u_1 and u_2 are velocities before impact, in the line of impact, and v_1, v_2 are velocities after impact. Using the fact that the total momentum is unchanged, superimpose (Fig. 3) the new energy diagram $AHKB$ on the old.

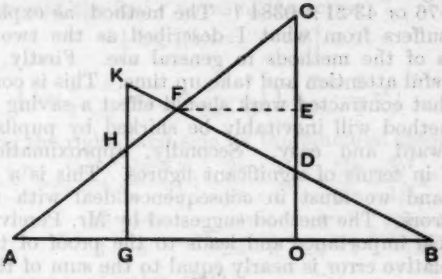


FIG. 3.

The loss of kinetic energy at impact

$$= \triangle FCD - \triangle FKH.$$

These are similar, and $\frac{HK}{CD} = e$, the coefficient of restitution, since $CD = u_1 - u_2$ and $KH = v_2 - v_1$.

Thus loss of kinetic energy

$$= (1 - e^2)\, \triangle FCD$$

$$= (1 - e^2) \frac{CD^2}{2(\cot C + \cot D)}$$

$$= \frac{(1 - e^2)(u_1 - u_2)^2}{2\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}.$$

The impulse received by each particle

$$= GO$$

$$= (1 + e)\, FE$$

$$= \frac{(1 + e)\, CD}{\cot C + \cot D}$$

$$= (1 + e)(u_1 - u_2) \Big/ \left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

C. G. PARADINE.

1093. *Further remarks on approximations.*

In Note 1074 (May, 1933), Mr. Percival gives an alternative way of correcting the error in a product which is suggested by the formula $(A+h)(B+k) \doteqdot AB \pm (hB+kA)$. This method is interesting, but of limited value when applied to examples in actual practice. It is valuable and instructive in examples of the type

$$1{\cdot}03 \times 1{\cdot}02 \doteqdot 1{\cdot}06,$$

but are we going to save time with the usual run of examples such as $37{\cdot}9 \times 8{\cdot}76$ or $43{\cdot}21 \times {\cdot}0384$? The method, as explained by Mr. Percival, suffers from what I described as the two serious disadvantages of the methods in general use. Firstly, it is apt to require careful attention and take up time. This is contrary to the principle that contracted work should effect a saving in time, and also the method will inevitably be shirked by pupils unless it is straightforward and easy. Secondly, approximation must be thought of in terms of significant figures. This is a fundamental principle, and we must in consequence deal with relative, not absolute errors. The method suggested by Mr. Percival is at least of theoretical importance and leads to the proof of the rule that the final relative error is nearly equal to the sum of the individual relative errors. This was left out of my article on approximations owing to lack of space.

I should also like to comment on Mr. Berry's remarks in the May *Gazette*. On p. 307 of my article (December, 1932) I advocated that an answer should contain the same number of significant figures as in the least accurate number used, not one or even two less as frequently advocated. Mr. Berry suggests that even the same number of figures may sometimes be one too few. I am entirely in agreement with that and also with his statement that "bad" $(n+1)$-figure accuracy may be better than correct n-figure accuracy. This is in accord with my remarks at the beginning of my article. It should be remembered that the extreme numbers in the group comprising n significant figures are in the ratio of 10 : 1, and some anomalies are bound to arise in a simple rule to embrace all cases. The rule is meant as a working rule. It could have been made entirely satisfactory by the addition of a qualifying phrase, but at the expense of simplicity. However, even extreme cases and all the points mentioned by Mr. Berry are really dealt with, if my article be considered as a whole, especially the parts on relative error. Thus on p. 309 of my article, Example 1 is $2{\cdot}68 \times 4{\cdot}12$. The answer is stated as 10·04 with error $\pm{\cdot}03$. In Example 2, the numbers in $34{\cdot}0632 \times 5{\cdot}26323$ are each contracted to four significant figures and a *correct* answer to *five* significant figures obtained. The examples on p. 315 might also be noticed.

I am not certain whether the method indicated by the equation $(x+\epsilon)(y+\epsilon) \doteqdot xy + \epsilon(x+y)$ is intended as a means of investigation or as a working method. The latter is similar to and suffers from the same drawbacks as Mr. Percival's method. Even in the most favourable case, it is not quicker than the method of relative error.

Thus in the example quoted by Mr. Berry, 90×120, the total relative error is immediately obtained as 1/100, which at once gives the same error which is obtained by the other method.

S. INMAN.

1094. *An approximation for* $\frac{1}{2}\log\{(1+x)/(1-x)\}$.

An interesting approximation for $\frac{1}{2}\log\{(1+x)/(1-x)\}$ was given by James Gregory in a letter to Michael Dary of the 9th April, 1672.* A rough draft of this letter may be seen in the Royal Society Manuscript Collection.

Let s_r represent the first r terms of the series

$$\tfrac{1}{2}\log\{(1+x)/(1-x)\}.$$

Then

$$\tfrac{1}{2}\log\frac{1+x}{1-x} = s_{n-2} + \frac{x^{2n-3}}{2n-3}\left(1 - \frac{2n-3}{2n-1}x^2\right)^{-1}.$$

Expanding the right-hand side by the binomial theorem,

$$\begin{aligned}\tfrac{1}{2}\log\frac{1+x}{1-x} &= s_{n-2} + \frac{x^{2n-3}}{2n-3}\left[1 + \frac{2n-3}{2n-1}x^2 + \frac{(2n-3)^2}{(2n-1)^2}x^4 + \ldots\right] \\ &= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2n-3}}{2n-3} + \frac{x^{2n-1}}{2n-1} + \frac{(2n-3)x^{2n+1}}{(2n-1)^2} + \ldots.\end{aligned}$$

Thus the error is of order x^{2n+1}.

A. INGLIS.

1095. *The arc of an ellipse.*

In the February number of the *Gazette* (p. 9) mention was made of early attempts to find the length of an arc of an ellipse. It was stated that " by applying the binomial theorem to the differential element of arc and integrating the series so obtained term by term, Newton obtained an expression for the length of a quadrant of an ellipse as the sum of two infinite series in 1676 ". This analysis appeared in Newton's famous letter to Oldenburg (and which was afterwards sent to Leibniz) of the 13th June, 1676, and it is given in the *Commercium Epistolicum* (London, 1712) on page 53. What is not so generally known is that a similar result was also given some years before by James Gregory (1638-1675), and is on page 25 of the same work. It forms part of a letter to John Collins, " the Mersennes of British Mathematicians ", and is dated 15th February, 1671. A rough draft of the letter in Gregory's writing is in the Royal Society Manuscript Collection.

Following the usual custom of the time, Gregory gave the result only, but the analysis is easily reconstructed when the work of the late Professor Gibson † and of Professor Turnbull ‡ is recalled.

Let the equation of an ellipse referred to rectangular axes through the centre be

$$\frac{x^2}{r^2} + \frac{y^2}{c^2} = 1.$$

* S. J. Rigaud, *Correspondence of Scientific Men of the Seventeenth Century* (Oxon. 1841), ii. 240.

† *Proc. Edinburgh Math. Soc.* (1), 41 (1922), 2-25.

‡ *Ibid.* (2), 3 (1933), 151-172.

Let P be a point on the ellipse whose ordinate is a, and let A be an extremity of the major axis. Then the arc AP is

$$\int_0^a \left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} dy = \int_0^a \left\{1+\frac{r^2y^2}{c^4}\left(1-\frac{y^2}{c^2}\right)^{-1}\right\}^{\frac{1}{2}} dy.$$

Expanding by the binomial theorem,

$$\begin{aligned}\text{arc } AP = \int_0^a \Big\{ & 1+\frac{1}{2}\frac{r^2y^2}{c^4}\left(1+\frac{y^2}{c^2}+\frac{y^4}{c^4}+\frac{y^6}{c^6}+\ldots\right) \\ & -\frac{1}{8}\frac{r^4y^4}{c^8}\left(1+\frac{2y^2}{c^2}+\frac{3y^4}{c^4}+\ldots\right) \\ & +\frac{1}{16}\frac{r^6y^6}{c^{12}}\left(1+\frac{3y^2}{c^2}+\ldots\right) \\ & -\frac{5}{128}\frac{r^8y^8}{c^{16}}(1+\ldots)+\ldots\ldots\ldots\Big\}\,dy \\ = a & +\frac{a^3r^2}{6c^4}+\frac{a^5r^2}{40c^8}(4c^2-r^2) \\ & +\frac{a^7r^2}{112c^{12}}(8c^4-4c^2r^2+r^4) \\ & +\frac{a^9r^2}{1152c^{16}}(64c^6-48c^4r^2+24c^2r^4-5r^6) \\ & +\ldots\ldots\ldots,\end{aligned}$$

which is Gregory's result.

A. INGLIS.

1096. *Stirling's Theorem and the Gamma Function. Continuity.* In my article (*Math. Gazette*, XVII, May 1933, pp. 114-121) I omitted to discuss the continuity of the Gamma function. With the definition I gave this is particularly easy. The function $\phi(x)$ is defined by an infinite series, whose remainder after n terms is $\phi(x+n)$. The inequalities established for $\phi(x)$ in §§ 2 and 5 show that this remainder tends uniformly to zero when n tends to infinity, if x is confined to any finite region. It follows that $\phi(x)$ is continuous at all finite points except those at which an individual term of the series becomes infinite, that is, except at zero or a negative integer; and the same is true of $\Gamma(x)$.

M. F. EGAN.

---

950. As [Plutarch] said of geometry, that if [a wife] be skilled in that she will not easily be a gamester or a dancer, may perfectly be said of religion.—Jeremy Taylor, *The Mysteriousness of Marriage.*

See *Conjugalia Praecepta*, xlviii, i.e. ii, 145 *c*:

τὰ δὲ τοιαῦτα μαθήματα πρῶτον ἀφίστησι τῶν ἀτόπων τὰς γυναῖκας. αἰσχυνθήσεται γὰρ ὀρχεῖσθαι γυνὴ γεωμετρεῖν μανθάνουσα, καὶ φαρμάκων ἅπτεσθαι οὐ προσδέξεται τοῖς Πλάτωνος ἁπτομένη λόγοις καὶ τοῖς Ξενοφῶντος.

REVIEWS.

The Distribution of Prime Numbers. By A. E. INGHAM. Pp. viii, 114. 7s. 6d. 1932. Cambridge Tracts, 30. (Cambridge University Press)

The distribution of the prime numbers 2, 3, 5, 7, 11, ... in the series 1, 2, 3, ... of all natural integers is in itself a matter of primary mathematical interest. Moreover, the present state of the subject is interesting: although greatly advanced by some brilliant discoveries and by much devoted and successful work, our knowledge of the distribution of primes is still incomplete on an essential point, and numerous attempts of two or three generations of mathematicians have failed to complete it. The most important definitely acquired result is the so-called "prime number theorem": the number of primes not exceeding x divided by $\operatorname{li} x$ tends to 1. (The function $\operatorname{li} x$ or "logarithmic integral" is defined by the (principal value) integral

$$\operatorname{li} x = \int_0^x (\log t)^{-1} dt.)$$

The unsettled essential problem that has withstood so many attacks is to find the true order of the "error term", *i.e.* of the difference between the number of primes not exceeding x and $\operatorname{li} x$. The distribution of primes is, strangely enough, intimately connected with the properties of an analytic function, the function $\zeta(s)$. This function is represented by the series

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \ldots$$

when the real part of s is greater than 1 and $(s-1)\,\zeta(s)$ is an integral function of s. The outstanding essential problem depends also on a property of $\zeta(s)$, and what so many eminent mathematicians have failed to accomplish is to decide whether the following theorem, stated by Riemann as a hypothesis, is true or not: *All non-real zeros of $\zeta(s)$ lie on* ONE *straight line.*

Mr. Ingham set himself the task of writing a summary of the subject, not only up to date but also accessible to a wider circle of readers. This task is certainly not an easy one; the theory is intricate, widespread and unfinished, and the author who has to select his materials, his methods and his form of exposition, is confronted with a threefold difficulty.

In selecting his materials, Mr. Ingham (very wisely, I think) concentrated his effort on the main points, on the prime number theorem and its error term; related subjects, although interesting in themselves, *e.g.* the distribution of primes in arithmetic progressions, are only rapidly mentioned or entirely discarded. The chapters are so arranged that they disclose gradually the relation between the prime numbers and the roots of the ζ-function. Chapter I shows how far it was possible to advance in the direction of the prime number theorem with the elementary methods introduced by Chebyshev, *i.e.* without the use of the ζ-function. Chapter II proves the prime number theorem, and shows that it is equivalent to the fact that the vertical line through the point $s=1$ is free from the zeros of the ζ-function. Chapter III shows that by assuring larger regions free from the zeros of ζ we can improve the estimation of the error-term. Chapter IV establishes the curious exact formulae connecting the primes and the roots of ζ, and discloses the relation between the order of magnitude of the error-term and the Riemann hypothesis. Chapter V deals with the sign of the error-term, and winds up by the most difficult subject, of which Mr. Ingham gives a complete exposition, with Littlewood's famous investigation of the error-term involving $\log\log\log x$.

The main points are developed with all necessary details; some less important or more difficult results are merely stated. I find that Mr. Ingham's choice of the materials was very lucky. But all choice in such an unfinished theory on which so much attention is centred is risky: had Mr. Ingham written his book a few months later, he could have included details on the work of Wiener, now only mentioned in Chapter II, because it has been much simplified

since by Landau; and he could have included a remark on the puzzling connection between the Riemann hypothesis and a certain "Gauss hypothesis" which has been discovered in the meantime by Deuring.

The selection of the methods was the second task of Mr. Ingham, and certainly not an easy one. The proof of the prime number theorem, for example, can be presented in a great variety of forms; and the author has to choose or rather to frame a variant which should be reasonably short, intrinsically simple, sufficiently typical and in harmony with the other chapters. I have the impression that Mr. Ingham has succeeded very well in fulfilling all these multifarious and delicate requirements.

The form of the exposition selected by Mr. Ingham deserves to be particularly noticed. The exposition of a paper printed in a periodical may be regarded as sufficiently good if the reader is put in a position to see that each step of the development is correct. Merely as much as this is insufficient for a book: the reader should also be put in a position to see that each step *serves a purpose*. The reader should even see, at least now and then, that it was possible to hit upon just this line of proof without a miracle. Mr. Ingham possesses, I think, a particular skill in writing clearly on difficult things. He distinguishes carefully the different sources of the proofs he presents (his distinction of two kinds of theorems by "numbering" and "lettering" may deserve imitation); he explains, without too much circumlocution, the chief motifs of a proof before embarking on details; and, generally, he does not aim at conquering the conviction of the reader by mere force or ruse or surprise, but prefers that it should be gladly given.

Mr. Ingham's book is the first available English book on the subject; but it may, and, I hope, it will, be read with much profit by readers of any tongue who are looking for an introduction to the subject or for a preparation for the more comprehensive treatises of Landau.

G. PÓLYA.

Krise und Neuaufbau in den exakten Wissenschaften. Fünf Wiener Vorträge. Pp. iv, 124. RM. 3.60. 1933. (Deuticke; Leipzig and Vienna)

To replenish the university chest at a time of economic stress by a course of popular lectures explaining that the most exact sciences too have had their moments of crisis and are not in a state of despair but are undergoing a process of rebuilding was an inspiring idea, and it was executed brilliantly.

Prof. Mark begins with a description of the shocks which refined experiment has dealt to classical physics in the course of the last fifty years. Prof. Thirring, who deals in the second lecture both with relativity and with the anomalies of particle and wave-mechanics, compares himself to the caricaturist who attempts only to suggest a character by means of a few salient strokes. Prof. Hahn's subject is the crisis of intuition, and he gives a clear and witty account of Peano's space-filling curve, Brouwer's three regions with a single frontier common to them all, and other constructions which show how far conception can outrun illustration. Prof. Nöbeling introduces the fourth dimension and curved space, in a series of brief discussions, each culminating in the answer to some definite question: What is three-dimensional Euclidean space in analytical geometry? What is the relation of four-dimensional geometry to experience? Why does the physicist call the space of experience curved? Prof. Menger brings the course to an end: common sense is not to be trusted even in logic, where plausible first principles have been found to lead to the most inacceptable conclusions.

The design of the course will be apparent. Each lecture fits well into its place, and each is a model of lucidity, although the style varies completely from one to another. Bibliographies with no pretence to being exhaustive will set the reader on the road to serious study. The course in Vienna was a great popular success, and in putting the material into print the lecturers have put a world of readers into their debt.

E. H. N.

Mary Everest Boole : Collected Works, I-IV. Edited by E. M. COBHAM; preface by E. S. DUMMER. Pp. xli, 1566. 60s. 1931. (Daniel)

It is a great delight to welcome these four volumes of writings by Mary Everest Boole. They contain a bewildering variety of essays, articles, short works, letters, but the further one reads the clearer it becomes that throughout there is a certain unity. They reveal the personality and gradual development throughout a long life of what might have been an outstanding mind to whom scientific method was not only the key to knowledge and understanding, but also the secret of happiness and a most satisfactory *modus vivendi*.

Mary Everest, later the wife of George Boole the mathematician, was born in 1832 and died in 1916. The currents of her mind ran in a network which covered a huge domain. As the preface puts it, "Mary Everest began early to think of herself as finding her natural aim in service to truth for the sake of humanity, and through her long life she never swerved in this purpose". Her versatility drove her into pursuing this aim very comprehensively. Thus it was only part of her life-work to attempt to show how education could encourage the scientific temper in the young so that the problems of politics, economics and ethics might be treated in a suitable manner. Another objective was the introduction of mathematical ideas into the education of quite young children. She was deeply interested in sociology. She was in fact driven by an uncontrollable impulse to seek for truth, even in what might seem to a hard-boiled modern the most unlikely places. Thus she undertook an exhaustive and no doubt exhausting study of Mosaism and the relation of the discipline of Israel to the thought and life of the modern world. She tried to give an exposition on lines acceptable to Indian thought of some of the modern problems of special relevance to India, such as racial antagonism and the clash of varying traditions. She endeavoured also to give an account to the outside world of the rather strange psychological views which were peculiarly her own, but which nevertheless fit in very well with the modern Freudian discipline. Finally, during her last years, Mrs. Boole was concerned with the scientific observation of her oncoming physical infirmities. The gradual coming of blindness and deafness was to her a source of never-failing interest and she studied, in accordance with the best traditions of scientific methodology, the problems of deliquescence of faculty, the substitution of means of expression as organs failed, and methods of meeting sleeplessness and discomfort.

The first and last impression that these volumes make upon the reader who has had enough experience of the world in general (as opposed to the specifically mathematical and scientific world) to be able to appreciate them, is of a mind which had always one and the same desire, namely to seek truth by means of the canons of scientific method, whatever the subject-matter which happened at the moment to be engaging her attention. Mrs. Boole was essentially a person with ideas of almost unlimited heterogeneity, but with one and the same *Methodik* throughout. This is a characteristic which one encounters but rarely, and it has its own charm. But we must realise that it brings its own dangers too. Much of these four volumes is of interest only as a record of an exceptional and an exceptionally gifted personality. The difficulty is that the subjects attacked belong to domains of thought at very different stages of development. But it is psychologically and also sociologically interesting to read her writings, and it has been very much worth while to put on record the gallant attempts of Mrs. Boole to bring an orderly mind to bear on subjects, many of them chaotic.

It should, however, be emphasised that some of the writings are of value in themselves. In this category we would include *Lectures on the Logic of Arithmetic* (1903) and *The Preparation of the Child for Science* (1904), both of them good logic, good science and good pedagogy. *Philosophy and Fun of Algebra* (1909) and *Some Master Keys of the Science of Notation* (1911) are also interesting and stimulating.

D. M. WRINCH.

Georg Cantor. Gesammelte Abhandlungen. Edited by E. ZERMELO. Pp. vii, 486. RM. 48. 1932. (Springer, Berlin)

Georg Cantor, the founder of the theory of sets of points, was born in 1845 in St. Petersburg and died in 1918 in Halle. His father was a business man and wished him to become an engineer; his mother came of a family highly talented in the arts. From a very early age Cantor, however, felt that his lifework would lie in mathematics, and after years of opposition from his father, he ultimately obtained his acquiescence and began mathematical studies in Zürich in 1862. On the death of his father he left Zürich and in 1863 sat at the feet of Weierstrass and Kronecker in Berlin. His ideas found in Weierstrass sympathetic understanding. Weierstrass encouraged the young student, and he was soon surrounded by a congenial circle of young mathematicians, though Kronecker had little sympathy with his standpoint, and H. A. Schwarz, who was two years older, later opposed him and did all he could to hamper his progress.

In 1869 Cantor became Privatdozent at the University of Halle, and under the influence of Heine, who was Professor there, became interested in trigonometric series. In this subject, Cantor had great successes: among his achievements was the proof of the uniqueness of a trigonometric expansion of a given function (*Crelle*, 72 (1870), 139-142, also reprinted in this volume). His work in this domain also influenced the development of his ideas in Sets of Points and in Transfinite Ordinals.

Cantor's greatest period stretched from 1871 to 1884. At the beginning of this period he happened quite accidentally to make the acquaintance of Dedekind. This was the beginning of a real friendship, which proved of the highest value both to the younger partner (Cantor was now twenty-seven years old) and to the senior partner. A series of thirty-eight letters survives, some of which are included in the volume under review. These throw a light upon Cantor's standpoint in his mathematical work and throw into relief the difference between the romanticism of Cantor and the severely classical outlook of Dedekind.

In 1874 Cantor married Fräulein Guttmann, and there were four daughters and two sons, no one of whom, apparently, has worked in mathematics. The year 1874 also marks Cantor's first publication on *Mengenlehre*. Although he had at first assumed that the continuum of real numbers was countable, he now used the theorem of the countability of algebraic numbers to prove that the number of the continuum is greater than $\aleph_0$. Another interesting milestone was reached when in 1878 Cantor published his proof that the number of a continuum is independent of its dimension number. This result struck Cantor (as it has also struck those who came after him) as very odd at first sight. He wrote to Dedekind, who helped him very much in this work, "je le vois, mais je ne le crois pas". In fact, there was very considerable delay in the publication of this work, as the results which it contained appeared to the referees to be so highly unplausible.

This delay was a source of great irritation to Cantor, not because it was in itself of any great moment, but because it was symptomatic of the enormous amount of opposition which he encountered from his colleagues. The deep importance of his work in *Mengenlehre* was understood only by a handful of people; the others who could not or would not understand did all they could to hamper him in his career. Honourable exceptions are Dedekind, Tannery, Mittag-Leffler, and apart from a short period of hostility, Hermite and Weierstrass.

In 1897 Cantor's publications come to an end and, as so often happens, just at this time came the recognition from the whole mathematical world of the value and importance of his ideas. In this year, in fact, the first International Congress of Mathematics took place in Zürich. This was a great occasion for Cantor. He had done much towards making it possible to inaugurate an

international society of mathematicians, and now after years of effort he was acclaimed by mathematicians of every kind as the originator of new ideas which had already led to "einer neuen Befruchtung der Funktionentheorie". It is good to know that this success brought great pleasure to Cantor, even though it had been so unfortunately delayed. Apparently the many awards he now received from universities and learned societies (including the Sylvester Medal of the Royal Society in 1904 and the honorary Doctorate from St. Andrews in 1911) also afforded him considerable satisfaction. His seventieth birthday, in 1915, was to have been the occasion of a great meeting of mathematicians, but owing to the war few mathematicians could get to Halle to do him honour. In 1918 he died.

This volume is a real storehouse for admirers of Cantor, a unique mathematician, who in spite of blind opposition from most of his colleagues succeeded in his own lifetime in obtaining recognition for important and deep new ideas in Theory of Numbers, in Algebra, in Function theory, in *Mengenlehre* and in the theory of infinite numbers, which have already had a great influence on the development of mathematical thought. It contains a reprint of the most important papers of Cantor, in particular of the "zwei ganz grossen und unsterblichen Arbeiten" (as Fraenkel describes them), "Über unendliche lineare Punktmannigfaltigkeiten" (1879-1884), and "Beiträge zur Begründung der transfiniten Mengenlehre" (1895-1897). It contains in addition a historical review of probability theory which Cantor gave to the meeting of the Naturforschenen Gesellschaft at Halle in 1873, a long article entitled "Mitteilungen zur Lehre vom Transfiniten", which he contributed to the *Zeitschrift für Philosophie und philosophisches Kritik* in 1887 and an interesting review of Frege's work *Die Grundlagen der Arithmetik* (1884), a selection from the correspondence of Cantor and Dedekind and finally an interesting life of Cantor by Fraenkel.

We would like to congratulate Professor Ernst Zermelo very heartily on his editorship and to offer him, on behalf of the many English admirers of Cantor, our most cordial thanks. D. M. Wrinch.

Vorlesungen über die hypergeometrische Funktion. By F. Klein and O. Haupt. Pp. ix, 344. Geh. RM. 22. Geb. RM. 23.60. 1933. Grundlehren der math. Wiss., 39. (Julius Springer)

The original edition of this book was written by Klein, and was founded on lectures delivered by him at the University of Göttingen in 1893-94. It has now been revised by Professor Otto Haupt of the University of Erlangen.

In the new edition the original arrangement of the work is adhered to. The book is divided into two parts, in the first of which are given the more familiar parts of the subject. This part consists of the historical development up to and including Riemann's work in 1857. First of all there is an account of Gauss' work, naturally including his well-known theorem on the value of the hypergeometric function with unit argument, and the continued fraction associated with the hypergeometric series. Then comes a discussion of the differential equation satisfied by Riemann's P-function and the 24 solutions of the hypergeometric equation. The third chapter contains an account of the representation of hypergeometric functions by definite integrals, while the last chapter of this part gives further work concerning Riemann's P-function.

The second part of the book opens with a discussion of the quotient of two particular solutions of the linear differential equation of the second order, and in particular of the equation satisfied by Riemann's function, leading to the Schwarzian derivative and conformal representation. The particular cases of Riccati's equation and Bessel's equation are considered in some detail.

There are applications to other subjects, such as spherical trigonometry and non-Euclidean geometry. These applications, which occupy a considerable

space, seem to be somewhat removed from the main part of the book, but they are typical of the sort of thing that Klein loved to introduce.

The book is one that could only have been written by a man like Klein. On the other hand, some of the methods used are probably not so attractive to many present-day mathematicians as they were to those of Klein's time. For example, while double-circuit integrals are used to represent the hypergeometric function, no mention is made of integrals of Barnes' type and their application to working out the relations between the 24 solutions of the hypergeometric equation. Probably, too, in an entirely new book, some space would have been given to more recent work on hypergeometric functions of several variables and to generalised hypergeometric series. Perhaps, however, such additions would not have been entirely in keeping with the spirit of Klein's work, although they would have added to the value of the book. W. N. B.

Functions of a Complex Variable. By T. M. MACROBERT. Second edition. Pp. xv, 347. 14s. 1933. (Macmillan)

This edition is very largely a reprint of the first, with the addition of four appendices: one elucidates and amplifies certain points in the main text, the second is concerned with the hypergeometric function and its asymptotic expansion, the third with some further properties of Legendre functions, including asymptotic expansions, and the fourth with the Fourier integral.

The author's aim is to establish Cauchy's theorem, to discuss the circle of ideas which immediately surrounds this theorem—the Taylor and Laurent expansions and the calculus of residues—and then to apply these ideas to special functions, of which the most important are the Gamma function, the Weierstrassian and Jacobian elliptic functions and Legendre and Bessel functions. There are many examples worked out in the text and an admirable collection for practice, numbering about 700. A student who has worked through the book should be able to handle that important tool, the contour integral, with skill and precision.

Naturally this is not an encyclopaedic treatise; for example, there is nothing about the maximum modulus principle, to mention only one topic of extreme importance in modern function-theory. We do not quarrel with the author for his silence on this point. But since he includes a paragraph on conformal representation, since he proves Weierstrass' theorem on the factorisation of integral functions and then stops just where the theory of these functions begins to be interesting, could he not have added some references for further reading? Even in English there are excellent treatises on these two subjects. Indeed the whole book would be more valuable if an appendix were added containing suggestions for reading and a guide to more advanced works.

The exposition is sound and generally clear. But the first chapter needs supplementing; is the first sentence, "A number of the form $p+iq$, where p and q are real and i is a root of the equation $i^2+1=0$, is called a *Complex Number*", quite fair as a definition? In Chapter IV, in evaluating several contour integrals, the method used in establishing Jordan's lemma concerning $\int e^{mis} f(z)\,dz$ taken round the whole or part of a "large" semicircle has to be employed each time. Would it not be an economy both of space and of thought to state and prove Jordan's lemma explicitly and then appeal to it in each of these examples?

Finally, the introduction of the functions considered in the later chapters is often a little forced. Surely the student could be told why he is to "consider" some unexpected expression, which, to quote G. B. Mathews' words on this very point, "might have dropped from the skies, or have been written on a wall by some spirit from another world".

This is undoubtedly a very useful text-book; we regret that it is not more inspiring. T. A. A. B.

Introduction to the Theory of Numbers. By L. E. Dickson. Pp. viii, 183. 13s. 6d. 1929. (University of Chicago Press; Cambridge University Press)

The theory of Numbers and the theory of Determinants enjoy a common facility in that each of them has been made the subject of a comprehensive history containing a report of every published investigation in their respective fields. For the theory of Determinants we are indebted to Sir Thomas Muir for a five-volume compilation (Macmillan), whereas the theory of Numbers has been dealt with in Prof. L. E. Dickson's three-volume *History* (Carnegie Institute). The book before us gives an introduction to the author's larger work in a form suitable to a reader without any special knowledge of the subject. After preliminaries, which sometimes seem irrelevant to a systematic study of the subject, the author gives an account of the Gaussian theory of quadratic forms, which would have been more satisfying by the inclusion of a section on the modular group. Dedekind's condition for the solubility of

$$ax^2+by^2+cz^2=0$$

in rational numbers is included. The book contains no mention of Algebraic Numbers, which shed much light on such matters as the composition of Forms. The book closes with an account of Thue's theorem (which only dates from 1909) and some of the deductions from it. This theorem states that if

$$H(x, y)=ax^n+bx^{n-1}y+\ldots+ky^n$$

is an irreducible polynomial of degree $\geqslant 3$, with rational integral coefficients, the equation $H(x, y)=C$ has, at most, a finite number of rational integral solutions.

We cordially recommend the book to a reader desiring to gain some of the central ideas of the subject. W. E. H. B.

Essai sur la développement de la théorie des fonctions de variables complexes. By G. Julia. Pp. viii, 53. 12 fr. 1933. (Gauthier-Villars)

This booklet consists of a lecture to the recent International Congress at Zürich. It is a description of the progress of complex variable theory from the time of Cauchy to the present day, and shows well the vast extent of the subject. The space at the author's disposal does not allow him to give any technical details, and so no indication is given of the difficulties which have been overcome in this subject. The book brings out how new fields of great generality have often been opened out by the search for new methods to attack special problems which were often suggested by, but resisted solution by, old methods. In addition to showing the extent of the subject, the book gives some idea of the relations between its different branches and indicates where it has been stimulated by the modern theory of the real variable.

The book is easy to read and can be recommended to any who, knowing something of part of complex variable theory, wish to have an outline, without details, of the present position of the subject. R. C.

Théorie mathématique de l'Assurance invalidité et de l'Assurance nuptialité. Calcul des primes et des réserves. By H. Galbrun. Pp. viii, 183. 45 fr. 1933. (Gauthier-Villars)

This work is the fifth section of the third volume of the *Traité du Calcul des Probabilités*, the fourth section of which has already been reviewed (*Gazette*, XVII, July, p. 221). Of the five sections that have appeared, four have been written by Dr. Galbrun, and these together contain over 900 pages; as this

may be continued, some doubt arises as to the advisability of including so much detail in a treatise of this nature. As our Samuel Butler says,

"For brevity is very good
If w'are or are not understood".

Now brevity in mathematics is not only good, but for clear thinking it is essential.

The author's desire to cater, at the same time, for the mathematician and for the student has rather tended to make the work a juxtaposition of simple complexities with complex simplicities. Thus starting on each occasion by expressing the compound probabilities in the form of an equation involving double integrals, simplifying by the use of Dirichlet's theorem and evaluating by quadratures, he seems suddenly to think that the student may not understand, and so informs him that

$$\int_{\beta}^{\infty} q(\beta, \beta, \gamma)\, d\gamma = 1,$$

as this represents the probability that a life aged β will die before reaching age infinity; the most suitable comment would seem to be that of Robert Recorde's scholar: "Sir, I do thank you much for this reason, and I trust that I do perceive the thing". But Dr. Galbrun has little faith, either in the student's perception or in his memory, and takes every opportunity of repeating similar warnings. Again, the length of the formulae is much increased by the method of writing the present value of unit amount due n years hence as $(1+i)^{-(x-\alpha)}$ instead of the almost universal v^n, and his form becomes still longer in the case of fractional terms. The modifications required in subdividing the unit interval might be left to the student, or, if given at all, one example should be sufficient.

From the nature of the work we must presume some knowledge of mathematics on the part of the student, yet we are given in detail the calculation of the Gregory-Newton quadrature formula; since this is well known and has been used in actuarial calculations for over a century, it seems a pity that several pages should be given to it here, with the coefficients c_1, c_2, c_7, c_{η} written in detail, each with the integral sign, and the numerical values of the first five, in fractions and decimals—and this in both parts 4 and 5.

The notation should also spare us the reiteration of such inequalities as $x < y < z$. It is not the lack of mathematics that is most felt, not the want of more and longer formulae, but of more and better statistics. We may wonder why, in the absence of the necessary data, such particularity should be given to exact calculations, and, in the interesting fourth chapter, we are partly admitted into the secret, for we read that in France, assurance companies are prohibited from using policies which grant waiver of premiums in case of illness, and from the account given we can draw the conclusion that the total guile of the French actuary is not less than that of his English and American cousins, and we are told how the difficulty is overcome by reassuring the speculative portion in some company which does a class of business not subject to the laws governing insurance companies. This leads to a transference of a portion of the premium from company A to company B, and a transference of the whole of the premium for the term of sickness from B to A in the event of the contingency happening; as the life may have been lost sight of, it is necessary to estimate what length of illness he should have incurred. We think, however, that the actuary of A would find what the figures of B amounted to, before using the longer formula as a weapon with which to intimidate his rival. De Morgan, in the *Companion to the Almanac for* 1840, gives an excellent illustration of condensation which might well be imitated. The problem with its solution is too long for quotation, but the preamble might be recalled with advantage.

"The following problem will include every case we have yet seen proposed of annuities, whether for the whole of life, or temporary or deferred, increasing or decreasing uniformly; and also of insurances: with every manner yet proposed of paying the premium. It matters nothing that it involves payment of premium after the benefits begin to be received; since every application of it will require the part of the premiums so paid to be made equal to nothing."

Chapter VI deals with the application of Tchebycheff's theorem to the estimate of the different accounts, and is one of the best in the book, circumstances having been too much for the author, so that he has been compelled to introduce single letters with subscripts to represent the longer integrals; this is an abbreviation which we wish had been made earlier. No account is given of the growing use of Stieltjes' integrals in the subject of insurance, an interesting account of which has been given by Dr. Steffensen in the *Journal of the Institute of Actuaries*, vol. lxiii.

In spite of the length of Dr. Galbrun's work as a whole, it is well worth reading by those who have the necessary time and patience, for it is clearly written, exact though exacting; but it would be hard to decide for what class of reader it has been designed. W. S.

Statistische Mechanik auf quantentheoretische Grundlage. By P. JORDAN. Pp. xi, 112. RM. 8.20. 1933. Die Wissenschaft, 87. (Vieweg, Brauschweig)

After an introductory first chapter giving a brief account of the classical statistical theory of an ideal gas, the book consists of two main chapters. Chapter II is headed "Quantum Statistics of Ergodic Systems", and succeeds in the space of less than seventy pages in giving a comprehensive rational account of the subject. It starts with the definition of discrete energy levels, and in formulating the fundamentals of the subject due account is taken of the Principle of Indeterminacy. The Darwin-Fowler method of partition functions is sketched, and is used as an aid in developing the theory as far as a discussion of the canonical (Boltzmann) distribution, and of entropy. There follows a series of important applications to the ideal gas, the equipartition of energy, specific heats of gases, the theory of radiation, and the thermodynamics of crystal lattices. Chapter III is on the theory of Degeneracy, or the part of the subject which takes account of the identity of the fundamental systems (electrons, or α-particles, etc.) in an assembly. By setting up a correspondence with an assembly of oscillators, or "quasi-oscillators", which can be treated by the methods of Chapter II, a highly convenient means is evolved for calculating the distribution law for an assembly obeying Einstein-Bose statistics, or the Pauli principle, respectively. Then by regarding the classical theory as the limiting case, a natural solution emerges to Gibbs' Paradox on the additivity of entropy. New light is also given on Nernst's Heat Theorem.

In restricting the applications of the general theory to those of a fundamental character, and in avoiding what he calls the "Epsilontik" of the mathematics, Professor Jordan has given a perfectly balanced concise account of his subject. A reader equipped with a knowledge of the elements of quantum theory, and with an appreciation of the significance of the various results such as should be gained from an ordinary study of thermodynamics and the kinetic theory of gases, will be able to gain from the book a sound grasp of the principles, methods, and scope of statistical mechanics. He should find it too an admirable preparation for tackling, say, R. H. Fowler's monumental work on the subject. But, more than this, the author is to be congratulated on achieving his aim of producing a rational system of statistical mechanics following the classical lines of development, but based entirely on the postulates of the quantum theory. W. H. MCCREA.

Causality, a law of nature or a maxim of the naturalist? By L. SILBERSTEIN. Pp. viii, 159. 6s. 6d. 1933. (Macmillan)

This book is an extension of a lecture given in 1932 in Toronto. Its main thesis is that the Principle of Causality is to be looked upon as a useful maxim in leading to discoveries about nature, rather than as being itself a law of nature. Put roughly, this means that if a scientist observes any effect, then he should look for a *cause*, not on account of its necessary existence, but because in so doing he will probably find out something which is of interest to him.

The value to science of such a discussion is, however, open to question. The object of science is to predict the results of certain observations, when the results of other observations are known. The machinery of this prediction consists of what we are accustomed to call laws of nature, together with mathematical technique. But we first have to discover of how much of our future experience it is legitimate to require a prediction. When we know how to deal with the whole of this, whether our results are stated as probabilities or otherwise, we have got all the "causality" of which science can take cognisance. It is unprofitable, and contrary to the scientific method, *a priori* to define a Principle of Causality, and then to examine whether nature obeys it.

The latter part of the book is devoted to criticisms of the Principle of Indeterminacy, which criticisms one cannot help regarding as loose and inconclusive. There can be little doubt that the introduction of this principle into physics marks a definite advance, being essentially just the recognition of the disturbance of a system by the act of observation. The current statement of the principle may sometime need revision, but there can be no return to the position held before its recognition. Nor can one see any scientific value in the author's suggestion that fundamental Determinism may be "retained as a heuristic maxim". Clearly science has no permanent place for mere maxims, nor for anything short of laws of nature.

W. H. MCCREA.

Network Synthesis. Synthesis of a Finite Four-Terminal Network from its Prescribed Driving-Point Functions and Transfer Function. By C. M. GEWERTZ. Pp. vi, 257. 23s. 1933. (Baillière, Tindall and Cox)

Following an introduction containing a statement of the problem and a review of previous work on the subject, the author shows in Part I that the impedances and admittances of a four-terminal network constitute the elements of two reciprocal square matrices. These elements are thereafter considered as functions of a complex variable λ, whose value along the positive imaginary axis is the frequency, multiplied by $2\pi\sqrt{-1}$, of a sinusoidal applied e.m.f. or current, in which case the elements are the complex admittances and impedances of ordinary A.C. theory.

In the next four parts are evolved the necessary conditions fulfilled when a matrix is constituted of either set of functions, impedances or admittances, of a finite passive linear four-terminal network.

Restrictions are imposed on the matrix elements, the determinant of the elements, and the determinant of their real parts, over the λ plane. The author terms any matrix "positive real" when governed by these conditions, and a general synthetic method is considered by the author to demonstrate that they are sufficient.

Part VI deals with the matrices associated with certain standard networks.

Part VII reviews certain formal properties of matrices, and shows how certain modes of network combination correspond to operations with the associated matrices, addition and multiplication in particular.

On this basis, the author develops his synthetic methods in the last four

parts. In every case a " positive real " matrix is split into the sum of simpler " positive real " matrices.

Part VIII deals with the case where poles of the elements are located on the imaginary axis in the λ plane, when the functions may be expanded in a series of partial fractions that form " positive real " matrices corresponding to standard reactive structures, owing to the restrictions imposed on the residues at such poles in the conditions. In this way the matrix may be completely resolved if all its poles lie on the imaginary axis, or reduced to one without poles there. A transformation enables this part to deal with the case where all poles and zeros lie on the negative real axis.

Part IX sketches a method of synthesis by direct splitting.

In Parts X and XI are detailed general methods of synthesis for a matrix cleared of poles on the imaginary axis. By splitting a matrix into the sum of two, one being of simple form, and the other having zeros on the imaginary axis for its determinant, poles on the same axis are produced in the elements of the reciprocal matrix, and this may be resolved as in Part VIII, giving rise to a matrix whose elements are polynomial fractions of lower degree.

The work closes with an Appendix containing examples, and a bibliography, but no index.

The specialist on electrical networks will be grateful for being provided with a powerful tool for his use. Those unfamiliar with network theory will not find the book very readable, in spite of the author's attempts to render it so by the expansion and elaboration of what was originally a thesis. This has resulted in more diffuseness than clarity. In addition, the author seems to favour " brighter mathematics ", to judge from his style. Such artifices may be a mental irritant instead of a tonic to the enthusiasm; it is so much a matter of temperament. It is perhaps safer to treat an involved subject with perfect formality.

S. W. R.

Rayleigh's Principle and its Applications to Engineering. By G. TEMPLE and W. G. BICKLEY. Pp. xiii, 152. 14s. 1933. (Oxford)

The late Lord Rayleigh showed (*Sound*, vol. i. arts. 88, 89) that, if a system of bodies, which has a number of possible normal modes of oscillation, is constrained in some way to oscillate, under its own potential energy, in a mode which is not the slowest normal mode of the unconstrained system, then its frequency in such constrained oscillation will be less than the frequency of its slowest normal mode. When the type or mode of oscillation is known the calculation of the frequency is usually a very simple matter, requiring only the calculation of the mean potential and kinetic energies of the system. The accurate solution of the problem of oscillation of a system with an infinite number of degrees of freedom requires the solution of a differential equation followed by the (usually) much more difficult problem of finding the roots of a transcendental equation. In a single example Rayleigh showed how a very good approximation to the slowest frequency can be found without even solving the differential equation—indeed, in his simple example of a violin string, without even writing down the differential equation. This is the great virtue of Rayleigh's principle, that extremely good results for the frequency of the slowest mode can be got by using the energy equation in which the correct transcendental function specifying the mode of motion is replaced by an extremely simple function of algebraic or other common type. The substituted function is supplied by guesswork and the error in the frequency is of second order as compared with the first order error in the function.

In many problems only the slowest frequency is needed—or, at least, the slowest frequency is of far the greatest importance. But in other problems the second or third frequency may be needed, and there was nothing in Rayleigh's method that showed how any but the smallest frequency could

be found. W. Ritz afterwards discovered a method, which was really an extension of Rayleigh's method, by means of which, but with much labour, other frequencies beyond the first could be found. This method of Ritz starts from the differential equation, and no conception of energy need come into the process. In this last respect the method is better than Rayleigh's because there are some questions, such as the oscillation of very thin plates, where Ritz's method would not give a true energy equation, but would probably give a better result than an energy equation would give.

In the book under review the authors, besides working out a number of problems in applied mathematics, have supplied the theory of Rayleigh's Principle. In fact, the two most important chapters in the book are devoted to the study of the functions arising in these vibration problems, and therein we are shown under what conditions Rayleigh's Principle will give good results. A further most important contribution to the subject is a method of successively approximating to the slowest frequency and of using the same set of calculations to give an approximation to the second frequency, the last being less accurate at each stage than the corresponding approximation to the slowest frequency. All this is done by a function here called an Induction Function, each problem having its own induction function. While this function is very useful in theory, nevertheless the authors very soon discard it in practice because, as it has always discontinuities, it leads to cumbersome integrals in the applications. These chapters on the rigour of the subject form a useful piece of work of which applied mathematicians themselves have felt the need. It is so easy, when we have found a simple trick that usually works and gives good results, to go on using it without thoroughly understanding it. It is generally only when it fails to work that we begin to investigate it properly—or turn for help to somebody who can do it for us.

Recognising the abstruseness of the mathematical arguments, the authors have done their best to illustrate the theory by simple applications, and have also given a résumé of the mathematical arguments and conclusions at the end of Chapter 4 for the benefit of those who are not exactly mathematical experts.

If the book impresses on engineers and physicists the usefulness and the simplicity of this method of solving difficult problems of vibrations and of elastic stability by means of very simple mathematics, it will have been very well worth the labour of production.

J. PRESCOTT.

Geschichte der Astronomie. By R. WOLF. Pp. xiv, 815. RM. 15. 1877, reprinted 1933. (Oldenbourg, Munich; K. F. Koehler's Antiquarium, Leipzig)

Koehler's Antiquarium has given us an exact reprint of Wolf's History of Astronomy. Nothing except the covers is new. The title-page is unchanged, the misprints and the instructions to binders, the type and index are as they were. There is no continuation to our days.

Even as it is, the reprint ought to be useful. Of the history of Chinese and Chaldean astronomy and of the influence of the former on Greek astronomy little was known in 1877. But these occupy only a small part of the volume. No careful history of astronomy from Copernicus to 1877 is easily antiquated, and even Zinner's *Geschichte der Sternkunde* (1931) does not replace Wolf's treatise, for Wolf's work is particularly rich in references, which Zinner systematically omits, and in personal details which are on the whole more enlivening than distracting. If Wolf's *Geschichte der Astronomie* has been superseded at all, it is by his own *Handbuch der Astronomie* (1890-3), but though the *Handbuch* is a fuller manual for the student than the *Geschichte*, the *Geschichte* is far more readable, and it is to be hoped that the reprint will gain readers for it. If only the price were less!

J. K. FOTHERINGHAM.

Gleichgewichtsfiguren rotierender Flüssigkeiten. By L. LICHTENSTEIN. Pp. vii, 174. RM. 15.60. 1933. (Julius Springer, Berlin)

The problem of ascertaining the possible figures of relative equilibrium of a gravitating mass of liquid, when rotating about a fixed axis, and of considering the stability of the figures, has given rise to a considerable literature, largely on account of the cosmological applications. The book under review is concerned almost entirely with the discussion of existence theorems, questions of stability being left for another monograph; and the emphasis is throughout on the pure mathematics of the subject—on the rigorous proof of the theorems. After two preliminary chapters, which contain a brief account of such matters as the relevant potential theory, Maclaurin's and Jacobi's ellipsoids, and certain easily proved general theorems on the figures of equilibrium, the author passes to the question, first considered in its generality by Poincaré, of the existence of equilibrium figures for a homogeneous fluid in the neighbourhood of a given figure of equilibrium. He passes on to the consideration of heterogeneous fluids, and in the last chapter deals with equilibrium forms in the neighbourhood of forms which satisfy the equations of relative equilibrium only approximately. In this last chapter we meet such old friends as Roche's satellites and the annulus discussed by Laplace and Mme. Kowalewsky.

The existence problems are throughout made to depend on the solution of a non-linear integral (or rather, integro-differential) equation. (The author's book on such equations was reviewed in the *Gazette* in 1932, Vol. XVI, p. 361; the present volume contains the applications and necessary extensions.) The work is very largely original, based on the author's own researches and those of his pupils. It is highly analytical throughout, more general than that of Liapounoff, more rigorous—and also more general—but much less easily understandable, than that of Poincaré.

The book is in paper covers. Its price is exorbitant. S. G.

Methods of Statistical Analysis in the Social Sciences. By G. R. DAVIES and W. F. CROWDER. Pp. xi, 355, with 44 charts. 20s. 1933. (John Wiley, New York; Chapman and Hall)

This book follows to a certain extent the standard lines of an elementary manual of statistics. The various chapters are devoted to: gathering and presenting data, averages, dispersion, index numbers, time-series analysis, correlation and probability and frequency curves. But it is emphasised throughout that the aim of the authors is to concentrate on laboratory methods and the logic underlying them, and that it is the particular domain of the social sciences that they are concerned with, as shown by the nature of the examples used in illustration of the methods. The book will not be very satisfying to the mathematical reader, for although mathematical notes are appended which deal with such elementary points as the rule of signs, the use of logarithms and elementary differentiation, the mathematics is not complete on such questions as the fitting of frequency curves, the theory of the binomial series, probable error and so on, but the reader is rather introduced to a standardised arithmetical technique. Some of the methods, *e.g.* the logarithmic frequency distribution and "first moment" correlation methods, belong to the authors' own school of statistics, and are not widely known in other places, which makes it all the more desirable that their mathematical basis should be clearly given. However, this book is not the place to give the demonstrations required, since it is addressed to non-mathematical readers, and these will find a considerable amount of useful information relating to the classifying and interpretation of business statistics. Numerous examples are given for working by the student. It is a pity that so many of these, and of the illustrative examples, are obviously "made up", and do not represent real data, a point realised by the authors,

who nevertheless hope that the emphasis which is given specific processes will offset these shortcomings. No attempt is made to go at all deeply into the question of sampling errors. Tables are added of logarithms, of powers, roots and reciprocals, and of the normal error curve, while a further table is reproduced for use in connection with the fitting of parabolic curves to data, and there is a useful graphic table of logarithms for those who like that sort of thing. J. WISHART.

Introduction à la Géométrie Infinitésimale Directe. By G. BOULIGAND. Pp. viii, 230. 36 fr. 1932. (Vuibert)

In the attitude of mathematicians to functions of a real variable, we can distinguish three periods. First, the properties which belong to functions of the simplest kind are tacitly assumed: we speak of the limit or the derivative of a function without knowing that our language implies restrictions on the function. Next, such properties as continuity are carefully analysed: recognising that they are not inherent in the idea of functionality, we nevertheless treat their absence as exceptional and confine our attention to functions which behave nicely almost everywhere; the advance is in a precise recognition of limitations we are imposing, not in any extension of our range. Lastly, instead of postulating, we investigate: four limits which must exist take the place of one which may not, and the theory of functions has come to birth.

Infinitesimal geometry shows a parallel development. First the existence of tangents to curves and tangent planes to surfaces, except at points that are obviously singular, is taken for granted. Then the relation of this assumption to the analytical specification of the curve or surface is examined; to this second period belong many of the results made available for the English reader in Fowler's Cambridge Tract, and the majority of investigations concerning Jordan curves. Professor Bouligand takes us into the third period, in which we study structure rather than existence. For example, instead of first defining a curve and then investigating conditions under which the line PQ tends to a definite limiting position as P and Q tend to a point O of the curve, we associate with any point O of any aggregate Γ the whole aggregate of limiting positions of the line, an aggregate called the paratangent of Γ at O. As anyone acquainted with the literature on the defining of a curve or surface will appreciate, this is an immense simplification. Moreover, it is found that if the paratangent of a plane continuum at a point O does not include every line through O, the continuum in the neighbourhood of O is a Jordan curve, and such theorems as this throw new light on the problems of the second period.

Not all of Professor Bouligand's book deals with concepts as recent as the paratangent. His subject includes the theory of sets of points in two or more dimensions, and his powers of selection, arrangement, and exposition are well exerted in his account of boundaries, convexity, the Cantor-Minkowski construction, limits of sets of aggregates, and similar matters. Many students who have felt the need for a simple introduction to this theory will fail to recognise from the title that it is being offered here, but it is to be hoped that curiosity, the author's reputation, and such publicity as reviews and references can give, will gradually bring to the book the success it deserves.

Although the table of contents is unusually ample, the book would be improved by an alphabetical index of definitions. E. H. N.

Darstellende Geometrie. By H. VON SANDEN. Pp. viii, 111, 42. RM. 6.40. 1931. Teubners mathematische Leitfäden, 2. (Teubner)

The subject of "plans and elevations" is usually delegated to the art-master in this country, though it is a help, only second to the use of models,

in grasping the relations of spatial figures. The book under review is illustrated with excellent diagrams, which are collected together into a pamphlet of forty-two pages slipped into a holder in the cover. Only the simplest principles of solid geometry and of elementary algebra are used, and these suffice to show how the various orthogonal projections of plane and solid objects may be obtained.

There are paragraphs on the sections of a cone, the intersections of cylinders, tubular surfaces, helicoids, and a final short treatment of perspective views. The aim throughout is practical, and the book would be of most use to those who teach students who, having a slender knowledge of solid geometry, desire to use that knowledge for some practical end in architecture or engineering.

H. G. F.

Aufgabensammlung zur projektiven Geometrie. By H. Timerding. Pp. 140. RM. 1.62. 1933. Sammlung Göschen, 1060. (Walter de Gruyter, Berlin)

The title of this book might easily mislead. It is not merely a collection of problems in construction, nor is it concerned with projective geometry in the strict sense, for Euclidean geometry is freely assumed throughout, and there is no attempt to place the subject on an axiomatic basis.

The book is divided into the following sections: introductory "problems", involutions, affinities, perspectivities, projectivities, collineations.

The part on affine transformations is specially interesting, and there is a very welcome proof of Pohlke's remarkable theorem that if any three coplanar intervals (no two collinear) be drawn from a point, they can be regarded as the parallel projections of three coterminous edges of a cube. This theorem, which has attracted the attention of such men as H. A. Schwarz, Reye, F. Schur and Klein, is here proved from the following: given any triangular prism, a section of it can be found similar to any given triangle. The proof given is simpler than those in Wendling's monograph on Pohlke's theorem, and the book is worth having for this reason alone.

Considering its size, the book covers a large amount of ground. Methods depending on cross-ratio are seldom used, and properties of a conic are often proved by projecting a figure involving a circle. Projective solid geometry is not touched, though spatial considerations are used to simplify proofs of properties in the plane.

This is a useful supplement to other treatments, and everyone who teaches projective geometry will find something in it to interest him. H. G. F.

Elements of Coordinate Geometry. By J. M. Child. Pp. xii, 468. 12s. 6d. 1933. (Macmillan)

The author is not satisfied with the ordinary presentment of the subject and has tried to write a Coordinate Geometry which shall be not merely a Conics. He has aimed at showing that coordinate geometry is not glorified graphs nor yet the lazy student's means of dodging the difficulties of pure geometry. We share his dissatisfaction and sympathise wholeheartedly with his aims, but feel that he has missed his target.

There are indeed some innovations in the volume. For example, there is a chapter on graphs of statistics which can be reduced to a straight-line law, and some useful emphasis is laid on equations of the form $L+kL'=0$; the positive sense of a straight line is defined as that in which x increases and straight lines are only allowed to make acute (positive or negative) angles with the x axis, so that the length of the perpendicular is, conveniently, always

$$-(ax+by+c)/(a^2+b^2),\ (a>0).$$

The explanation of gradients is neither concise nor clear, and it might well

have been omitted. It is in fact assumed a few pages later that the student has some knowledge of Differential Calculus, and, judging by some statements about points which ultimately coincide and chords which become tangents, it is apparently assumed that he will have acquired this knowledge in the nineteenth century. Nor is the treatment of "imaginary" points and points at infinity exempt from all the criticisms applicable to older books.

The general conic is discussed by solving the equation for x in terms of y and for y in terms of x. There are separate chapters on the three conics in which "the main idea is not the mere acquirement of the forms of the equations of the tangents and normals, etc., but the use of these equations to obtain geometrical facts about these curves". Thus we are given a proof for the ellipse of the property $CV.CT=CP^2$, by what is perhaps the best available method when projective geometry, orthogonal projection, ordinary geometry, and oblique axes are barred; later we have it again for the hyperbola. This is because the author wants to show that coordinate geometry "stands on its own feet" and "is no royal road for the lazy"; there is an apology in the Preface for the 80 pages given to conics, but actually at least twice that amount is devoted to them.

Tangential methods are not even used for the foci or for confocal conics. The usefulness of tangential coordinates and equations is "undeniably proved" in an example, but unfortunately that is on page 434. The tangential equation of the general conic is found, in effect, on p. 113, but instead of examples to illustrate its use, we are shown an unusually good contrivance for making a mountain $(24156m^4-82740m^3+103559m^2-56140m+11100=0)$ out of a molehill (the condition for $y+5=mx+6m$ to touch

$$2x^2-5xy+4y^2-3x-5y-9=0);$$

the alternative method on the following page shows how the redundant factor $(6m-5)^2$ might have been avoided, but it will leave the average student in the dark about the procedure to be adopted for, say,

$$(m+2)x+(3m+4)y=5m+6.$$

In the introduction to homogeneous coordinates, preference is given to trilinears rather than areals, for which no sufficient reason is assigned. Conditions such as that for perpendicularity are found by Cartesian methods and the author has little regard for symmetry.

There is a sprinkling of small errors, *e.g.* Joachimsthal is sometimes Joachimstahl, and Simpson's rule is quoted as Simson's; also Archimedes is accused of having set fire to the *Greek* ships. There is an interesting mistake in the condition for the origin to lie in an acute angle between the lines $ax+by+c=0$ and $a'x+b'y+c'=0$, which is stated to be

$$c^2/(a^2+b^2)+c'^2/(a'^2+b'^2)>2cc'/(aa'+bb');$$

the true condition is that cc' and $aa'+bb'$ should have opposite signs, which the author says is not *necessary*; actually his own condition is not *sufficient*, as may easily be verified by the example $2x+2y+1=0$, $x+1=0$. It will be a good puzzle for the reader of the book to spot the mistake in the argument of the text.

We therefore consider that the book is not to be recommended for its accuracy, for its style, or for its modern methods. We concede that the author has proved his thesis that coordinate geometry is no royal road for the lazy, or at least that this is true of Mr. Child's coordinate geometry. A. R.

Analytical Geometry of the Straight Line. By W. M. Baker. Pp. i, 63, v. 1s. 6d. 1933. (Bell)

"This book has been issued primarily to meet the needs of schools preparing for the Higher Mathematics papers in the Leaving Certificate examination of

the Scottish Education Department, but it should also prove acceptable in other quarters where a knowledge of the Analytical Geometry of the Straight Line only is required."

It is a shattering thought that there are "quarters" where such a limited range of knowledge is required for examination purposes, and that too in a Higher Mathematics paper. It is true that as a rule it is only possible to add the analytical geometry of the circle, but that at any rate gives the teacher some kind of opportunity to tell the student what analytical geometry is about. It is entirely misleading to concentrate on the straight line. But if an examining authority insists on such an odd requirement, such a book as this is the inevitable result! It only remains to add that the subject-matter is well presented, there are plenty of examples, some revision papers, and a set of answers. N. M. G.

Lebendige Mathematik. By F. AUERBACH. Pp. 355. Geh. RM. 7.00; geb. RM. 9.00. 1929. (Hirt, Breslau)

There are many styles of mathematical writing, ranging from that of formal logic to those of the best and worst of "popular" expositions. In England we have the tradition of De Morgan; we have Whitehead's brilliant historical and philosophical writings; and we have been shown by Eddington how the most abstruse subject can be made both intelligible and delightful. But no modern English writer has given us a purely descriptive and comprehensive account of mathematics as a whole such as we find in this extremely readable, well-printed and copiously illustrated book. Those who pursue the straight and narrow path of strict mathematical form will perhaps not care for its comparatively gossipy style, for they will find in it no rigorous argument and very little logical order: but it is well that some mathematicians should so far break their isolation from the world as to write books about their subject which may be read by intelligent non-mathematical readers, presenting it as being of some human significance, some general human interest.

The author realises how many people are possessed by a fear of mathematics, and he sets out to dispel fear by knowledge, the only true way. Following a main division which he describes briefly as "Schauen und Denken", he gives us first a geometrical section, in the course of which he describes all the simpler lines, surfaces and solids that one meets in elementary or advanced mathematics, with interesting discussions of curvature, perspective, infinity, the fourth dimension, aesthetic values; and such a wealth of apt illustrations as any teacher of mathematics might envy.

After a short discussion of time, by way of interlude, he proceeds to Number. He discusses the various classes of numbers, lingering somewhat over the fascinations of primes, and gives us, still in readable form, an introduction to the ideas of series and functions, with due attention to such matters as continuity and convergence. He goes on to deal with the calculus, and with equations, both algebraic and differential, winding up with chapters on statistics and on the diverse applications of mathematics.

The book is a development of a course of lectures and should prove useful to anyone who has to talk about mathematics to a non-mathematical audience. The schoolmaster lecturing on mathematics to a classical sixth form might do worse than follow it. His listeners will certainly be interested and will carry away a good impression of the nature and scope of the subject: but they may want to know more about some matters that are only briefly mentioned, such as, for example, conformal representation, the normal curve of errors, and map projection. (On p. 197 a geometrical construction is given for Mercator's projection, a surprising mistake in an otherwise carefully written book.)

The English reader with some acquaintance with mathematics will find that a quite modest knowledge of German will enable him to read most of these chapters: the effort he expends will be amply repaid. E. H. L.

An Introductory Course of Mechanics. By E. G. PHILLIPS. Pp. viii, 255. 10s. 6d. 1933. (Cambridge)

Two short quotations, one from the publishers' announcement and the other from the author's preface, indicate the character of this book: "the author regards the subject throughout as a branch of Mathematics, not as an ancillary to Physics or Engineering", and "this book aims at developing the subject of Elementary Mechanics in as logical a manner as the nature of the subject permits".

It is rather puzzling to know for exactly what type of student this one-year course is intended. The author says that no previous knowledge of the subject is required, but a beginner will find the work difficult and the course long. The book is very different from most elementary treatises on mechanics in general use, as no appeal is made to experiment. Indeed, many will regard it as reactionary. But those who require a logical treatment of the subject will find here an excellent book. The text is clear and convincing, the worked examples are well chosen (although not very numerous), and the subject-matter covers an unusually wide range.

For the most part the treatment follows the conventional lines of Newtonian dynamics. Vector methods, including unit vectors and differentiation of vectors, are used freely, and a working knowledge of calculus is assumed. The usual theorems on the angular momentum of a system of particles are included. The second half of the book is devoted to statics and hydrostatics, forces in equilibrium being introduced as forces which produce no acceleration.

The book contains a large number of examples for the student. Few of them are very easy, and many are of the type which are set in examination papers for the final examination for the B.Sc. degree.

It need scarcely be added that the printing is all that could be desired.

R. O. S.

Elementary Mechanics of Solids. By H. A. BAXTER. Pp. x, 281. 8s. 6d. 1933. (Blackie)

The author in his preface states that the book is intended to be a comprehensive text-book, suited to beginners, of the elementary stages of mechanics, and to cover completely such courses as those for the Higher School Certificate Examinations. Rather more than half of the book is taken up with examples (practically none of them worked out), and the text contains an exposition of the theory of dynamics and statics starting from straight-line motion and triangle of forces, and finishing with three-dimensional motion of a body and stability of equilibrium.

The form of the book is reminiscent of the continental Course of Mechanics rather than of the average English text-book. Within each of the seven chapters the argument follows a continuous thread unbroken by sets of illustrative examples. These are grouped together at the ends of the chapters, and are really imposing in number. For the most part they are perfectly straightforward and quite simple.

The author bases his treatment on the Laws of Motion, although he appears at first sight to restrict the Third Law to impulses. A somewhat unusual nomenclature is introduced; the reluctance of a body to change its velocity is called its inertia, the British unit of inertia is called the pound, and the amount of inertia of a body is called its mass. Force is defined as the quantity mf, so that the units used throughout are absolute and there is no trouble with g. Problems in statics are first treated as those of forces which do not produce accelerations. Later a somewhat different concept of force is introduced, but its description on page 203 as a succession of vanishingly small impulses is not a happy one.

At the outset the author emphasizes the distinction between the scalar quantity "speed" and the vector quantity "velocity". Except in an

illustrative example the important graphical method of dealing with motion whose acceleration varies in magnitude is neglected, and even there the question of units, which often presents much difficulty, is not discussed. Elementary calculus methods are used from the start, but the statement

$$\frac{\delta v}{\delta t}=\frac{\delta v}{\delta s}\cdot\frac{\delta s}{\delta t};$$

therefore when δt vanishes we get $f=v\,.\,dv/ds$ comes as a shock. The proof of the formula $s=ut+\frac{1}{2}ft^2$ is unusual and not easy to follow, involving as it does the idea of extremely short intervals of time at each of two instants. There is no mention of the equally important formulae

$$s=vt-\tfrac{1}{2}ft^2 \quad \text{and} \quad s=\tfrac{1}{2}(u+v)t.$$

The latter, which is proved so easily from the speed-time curve, provides a most convincing illustration of the restricted validity of these results.

The treatment of the dynamics of a particle is unusually good. The author equates "cause" and "effect", that is, forces appear on one side of the equation of motion and mass × acceleration on the other side. In consequence there is never any difficulty with signs and no introduction of centrifugal forces and other confusing devices. It seems a pity that the same method has not been used when dealing with the motion of a lamina. There much stress is laid on the instantaneous centre of rotation. Throughout the book the importance of momentum as a physical quantity is emphasized.

The treatment of the subject is anything but hackneyed, and it is surprising how much freshness the author has introduced into such a well-trodden field. Few will be able to read the book without getting many new ideas. Here and there are small lapses, but in general the style is good and the meaning clear. The printing and the diagrams are excellent. In some places, particularly in the examples, a little more care might have been taken with regard to the use of fractions and the vinculum.

R. O. S.

Cours de Mécanique Rationnelle. II. Dynamique des systèmes matériels. By J. Chazy. Pp. vi, 461. 80 fr. 1933. (Gauthier-Villars)

This volume is the second part of a course of mechanics delivered at the Faculty of Science at Paris, the first part, *Dynamique du point matériel*, having been published a few months ago. It deals with the motion of a rigid body in two and three dimensions, the principle of Virtual Work, the equilibrium of a string, Newtonian potential, hydrostatics and hydrodynamics.

Following the traditional French practice, the author treats the subject with considerable generality, and in the preface he almost apologises for the introduction of a few simple illustrative problems. There are of course no examples for the reader. A striking illustration of this difference between British and continental text-books is found in the chapter on hydrodynamics: here it consists entirely of general bookwork, and not a single result is obtained.

A large part of this book is concerned with subjects which are usually considered difficult, but the author presents the matter in a way which makes everything seem simple and straightforward. The long chapter in which he gives the theory of the motion of a solid about a point (Poinsot's motion) and the motion of a symmetrical solid about a point on its axis (motion of a top) is excellent. For the most part the treatment of three-dimensional motion is based on moving axes and Euler's equations. The equations of Lagrange and Hamilton are established, and the former are used in the usual way to deal with small oscillations with two degrees of freedom.

It is to be regretted that the price of the complete work, 150 francs, is so high that few copies of this interesting and useful course are likely to be purchased by British readers.

R. O. S.

La Géométrie à la portée de tous. Pp. 117. 20 fr. 1931.

L'Arithmétique à la portée de tous. Pp. 96. 25 fr. 1932.

L'Algèbre et la Trigonométrie à la portée de tous. I. Calcul algébrique et équations. Pp. v, 56. 15 fr. **II. Étude de la variation des fonctions.** Pp. v, 43. 15 fr. 1933. By J. Poirée. (Gauthier-Villars)

The introduction to the first of these interesting books may be taken to give the aim of the author in writing all of them. He says that he has had in mind the beginner who knows no arithmetic, the schoolboy who finds it difficult to solve problems, the schoolmaster who has to teach the science—"so repellent and yet so useful"—and finally everyone who wishes to complete the studies of his youth. The first two in this list would find the discipline of the books too severe; the other two will find the books valuable. The books would serve also for rapid revision before an examination in elementary mathematics; they might consolidate the student's thought so that he saw each subject as a whole.

The book on geometry may be taken as an example of the scope of the author's plan. It begins at the very beginning of the subject and then gives most of what it is necessary to say about triangles and parallels, similarity of figures, the circle, polygons in a circle, various loci, the elementary properties of the ellipse, etc., concluding with chapters on lines and planes in space and the geometry of the cone, the sphere and the cylinder. All this is dealt with in a hundred large octavo pages not closely printed.

A few special points may be noticed. One thinks on reading the first pages of the "Arithmetic" that French children may find the earliest steps easier because they are already familiar with such words as *dizaine*, *centaine*, etc., as well as with the names of the natural numbers themselves. There must also be an old-fashioned flavour about these early studies in France. They still have problems about the "courier", quite in the style of our own books of a hundred years ago. And in another place we are reminded of the chapter on "Alligation" in those old books.

At the end of the "Arithmetic" is an interesting method of approximation in calculating square roots. If the square root of n is to be found and x is a first approximation, where $x^2 = n + R$, a second approximation is x', where

$$x' = x - \frac{R}{2x};$$

x'^2 is then equal to $n + R'$ (where $R' = R^2/4x^2$); a third approximation is x'', where

$$x'' = x' - \frac{R'}{2x'},$$

and so on.

The second of the books on algebra has a good description of the variation of functions, using curves to illustrate it, and thus introducing the elements of the calculus. The third part, on trigonometry, has not arrived. It is worth noticing, too, that in the "Geometry" the author defines the "director circle" of a conic as a circle whose centre is one of the foci and whose radius is equal to the transverse axis. He uses this circle for the construction by which he obtains the points of intersection of a straight line with a conic and the tangents to a conic from an external point. The name given to this circle is not in accordance with English usage.

The author keeps in view all through the aim mentioned above; there is no attempt at modern subtlety, he moves his triangles about and superposes one on another without any qualms of conscience. The figures in the books might be drawn better, perhaps; and there are a few misprints. But it is a *tour de force* to have put so much into so small a space. T. M. A. C.

Logical Sequence in Plane Geometry. Rapid revision by means of ideograms. By E. A. PRICE. Pp. 12. 1s. 6d. 1933. (Cambridge)

There is undoubtedly in some quarters a desire to bring order into the chaos at present existing in the logical presentation of geometrical theorems. Mr. E. A. Price, whose *Arithmetic*, produced in collaboration with the late C. Godfrey, is so widely used and appreciated, has devised a diagrammatic scheme for the systematic revision of elementary geometry, which will be a most acceptable guide to those who teach the subject and are frequently embarrassed by the different sequences followed in the many text-books at present in use. This scheme of so-called *Ideograms* shows at a glance how all the principal theorems are deduced from the properties of parallels and congruence, and at the same time indicates the additional theorems, etc., which are not essential to the logical sequence. Many teachers recognise that their pupils often have but a hazy idea of this thread of logic, and will welcome an aid of this nature in clearing up the confusion of geometrical notions that is frequently apparent in the minds of so many boys.

In addition to this main object, Mr. Price very properly insists most strongly on the important function of geometry teaching in giving opportunities for precise thinking and its concise expression. To define a geometrical conception accurately without redundancy is a very difficult art, and it is precisely this ability to " say what you mean and mean what you say ", which is so admirably tested and fostered by the proper teaching of geometry. A pamphlet has been prepared to accompany the sheet of Ideograms, and this gives in the most precise manner possible the definitions, axiomatic assumptions and theorems involved, so that the pupil may at any time realise exactly what he is supposed to have learnt and the best way of expressing this in words. Given the proper foundation as laid down in the pamphlet, the whole should prove an admirable method of systematic revision. In smaller type are hints and warnings to elucidate the main text, together with references to the most common misconceptions in the mind of the pupil, and—dare it be hinted ?—sometimes also in that of the teacher.

C. E. KEMP.

Elementary Algebra. I. By A. W. SIDDONS and C. T. DALTRY. With answers, pp. 132, xxii. 2s. 3d. Without answers, 2s. 1933. (Cambridge)

A generation ago educational reformers revolted against the mechanical and stereotyped methods of instruction then in vogue. The direct method of language teaching was introduced. Its counterpart in the teaching of algebra appeared in Godfrey and Siddons' *Algebra*, which led off with translation—the translation of the phrasing of a problem into an algebraical equation—and introduced the grammar of the subject step by step as the subject developed. To this method history had pointed the way. From Diophantus to Vieta algebraists had been concerned with the solution of problems and consequently with the solution of equations. To this method too the natural fascination of the puzzle beckoned invitingly.

In this approach to the subject the pupil is led to regard x as having one predetermined value which it is his task to discover.

In the formula method the emphasis is on the relation of several interdependent unknowns, none restricted to a particular value. Sir Percy Nunn was the protagonist of the method and Durell's recent *Elementary Algebra* one of its many exponents.

The text-book under review fuses the two methods. Retaining the good points of Godfrey and Siddons, it introduces much new matter chosen and handled with sure judgment. It appeals to the pupil's arithmetical experience. It deals patiently and clearly with the difficulties he encounters and the errors to which he is prone. It aims at giving him a thorough comprehension of what he is about and leaves no excuse for cramming or slipshod teaching.

Special mention should be made of the excellent treatment of graphs and of the frequent use of geometrical material for algebraical problems. And if a reviewer may be permitted such unprofessional sentiments, I should like to confess to feeling an enthusiasm for a book which seems to me to be the ideal towards which the research of the last twenty years has been reaching and to feeling that it is presumptuous of me to commend a book which commends itself on every page. F. C. B.

A Senior Algebra. By D. LARRETT. Pp. 308. Without answers, 3s. 6d. With answers, 4s. Or in two parts separately. (Harrap)

A very attractive volume, with good type and type-setting, with sets of examples at short intervals and plenty of revision questions. The explanatory text is concise but adequate. The book should prove comfortable to use in the classroom.

Part I (164 pp., with answers and logarithm tables) opens with a chapter on the roots of the quadratic equation. The results $\alpha+\beta=-b/a$, $\alpha\beta=c/a$ are derived from the general solution of $ax^2+bx^2+c=0$. This leads the pupil's ideas about functions of roots into a cul-de-sac, and it is a pity it should be given prominence. The usual method is given as an alternative.

Logarithms are introduced by powers of 3; no graph is given. A more complicated example than those given of the use of logarithms in numerical computation would have shown better the advantages of the lay-out used.

The chapter on series deals with Σn^2 and Σn^3 as well as arithmetic, geometric and harmonic progressions. k solves all questions in proportion and variation. All who dislike loose English and all who shudder when they hear that x^n means x multiplied by itself n times, will wince when they read the words "Notice in particular that such a ratio describes the number of times one quantity is greater than another, . . . ". The binomial theorem is treated inductively, and the proof by mathematical induction for a positive integral index is given in small type.

Part II (100 pp. and answers), serving as an introduction to the calculus, is more individualistic in treatment than Part I.

Chapter VII introduces the symbol $f(x)$ and revises graphs of simple functions. Chapter VIII deals with gradient and is freely illustrated by graphs. Chapter IX is devoted to maxima and minima. The graph of a function and its first derivative are sketched so as to show the correspondence between points showing maxima and minima on the one with points where $dy/dx=0$ on the other. And similarly the criterion for a maximum or minimum and the introduction of the point of inflexion are explained with the help of a three-fold graph showing the function and its first and second derivatives. Chapter X deals with rate of change. Chapter XI is headed "Wallis' Law", and uses Wallis' method of finding areas to introduce the idea of integration. Chapter XII deals with the definite integral based on the summation

$$\sum_{r=1}^{r=n} f(x_r)\,\delta x_r.$$

F. C. B.

Mathematische Spiele. By B. KERST. Pp. vi, 90. RM. 2. 1933. (G. Grote, Berlin)

In spite of the title there is not much mathematics and there are not many games. But a limited field of mathematical amusement is treated with true Teutonic thoroughness.

Starting with the 4-order case of Euler's problem "to place 36 stones, 6 of each of 6 colours, each set of six displaying the digits 1 to 6, in square formation so that no colour and no digit is repeated in any line or diagonal", the

author leads up to the solution of some magic squares. Thence he reaches magic star polygons. The solution of the hexagonal case proceeds for 14 pages. After that the reader might like to have a go for himself. But the author is pertinacious, he devotes another 5 pages to the 7-pointed star.

Then follow games of moving counters, of which the 15 stones in 16 cells is the most familiar. These are dealt with as cases of nets of polygons and linked up with the question of unicursal routes. The last chapter is devoted to a description of puzzles made of two similar pieces of bent wire, which have to be separated.

There are parts of the book very suggestive to puzzle-solvers who have been content with a " hit and miss " method. But no one keen enough to desire to tackle a puzzle systematically needs or would care for the elaboration of method displayed here. F. C. B.

Spiel und Spielzeug im Physikunterricht. By G. Dussler. Pp. 79. RM. 2.24. 1932. Beihefte der *Unterrichtsblätter für Mathematik und Naturwissenschaften*, 13. (Otto Salle, Frankfurt)

Physics is prolific of simple experiments and toys which illustrate very effectively underlying laws and phenomena. Scientific toys form an excellent means of stimulating curiosity and interest in students of all ages, and hence call forth eager demands for explanations.

A casual inspection of the book under review is sufficient to illustrate these facts. It consists of a comprehensive collection of descriptions of toys and easily constructed pieces of apparatus which illustrate physical principles in all branches of the subject. Brief explanations are given in most cases, many of which are illustrated by drawings.

Some of the examples are much more toys than illustrations of scientific facts, and the book would have gained in clarity from a stricter selection of material, a more definite classification into subjects and a fuller treatment of the more advanced subjects such as surface tension, the phenomena of moving fluids, and optical effects and illusions. Simple mechanical effects (inertia, centre of gravity, etc.) are included, and though the book is not intended to deal with mathematical problems, it seems a pity that more space is not devoted to the interesting subject of tops.

The book is full of interest to teachers of experimental physics. A long list of references, and a full name and subject index are appended.

L. G. V.

The " Omnes " Slide Rule. 3s. (Jackson Bros., Leeds)

This slide rule, which is just over 11 inches long, is constructed entirely of metal with the addition of an unbreakable transparent " free view " cursor. The markings or scale divisions are shown clearly in black upon white (or cream) enamel.

In the more common Mannheim slide rule we are accustomed to the scales A, B, C and D—scale A similar to scale B, and C similar to D, with A and D on the stock or body of the rule, B and C on the slider. Also when the ends are in line, the readings upon C and D show the square roots of the readings on A and B. On the " Omnes " rule the scale B is replaced by another scale similar to C but called scale R or the " root scale ", and upon this square roots are read directly and the fourth roots almost so. To facilitate calculations upon the circle, scale A is marked in advance of the usual initial I to allow the scale R, upon which the diameter d is read, to be set quickly at 0·7854 on A to give the area $(=\frac{1}{4}\pi d^2)$ or conversely to find d given the area. The reverse side of this slide has the usual sine, tangent and logarithmic scales for use in the ordinary way.

To determine cubes and cube roots a second slide is available for use with scale D. This scale F is graduated to show the cubes of the numbers on D

On this second slide also is provided the log.-log. scale, a B scale which is marked both L and KW, a scale marked Dynamo Motor and a scale marked Volt Loss. The Dynamo scale and the Motor scale are used in conjunction with the scales A and B, renamed for convenience HP and KW respectively, for determining efficiencies of dynamos and motors, while scale L and the Volt-Loss scale are for use with A, which gives the area of cross-section of the lead in square millimetres.

Very detailed instructions are issued with the instrument, and no average user should find difficulty in following through the various uses. A note issued with the instructions says: "If you permit yourself to be influenced by your first impression of it, you will probably return the 'Omnes' slide rule straight away". This is certainly true. It cannot be called an elegant instrument, but when one regards what it does and then looks at the price one must admit that the makers give value for money and have brought the slide rule within the range of the pocket of the ordinary student. E. J. A.

Application des imaginaires au Calcul vectoriel. By J.-B. Pomey. Pp. vi, 70. 20 fr. 1933. (Gauthier-Villars)

M. Pomey has collected a number of applications of the complex variable to geometry and electro-technics; there are pleasant sections, for example, on the Apollonius locus, on Savary's formula for the curvature of a roulette, on Cayley's rotation formula. Unfortunately, anyone looking for these matters would be completely misled by the title of the book and would ignore it. There is scarcely anything in the book to justify the "Calcul vectoriel" of the title; for the author, vectors are complex numbers, and if we use z_0 to denote the complex number conjugate to z, then we can, if we like, call $\frac{1}{2}(zu_0+uz_0)$ and $(zu_0-uz_0)/2i$ the scalar and vector products of the two vectors z, u. And then, in an application, when one of these expressions turns up, we have the vector calculus!

An attractive book—why was it not given a sensible title? T. A. A. B.

Höhere Mathematik für Mathematiker, Physiker und Ingenieure. Teil IV, Heft 2. By R. Rothe. Pp. 53-104. RM. 2. 1933. Teubners mathematische Leitfäden, 34. (Teubner)

This part of Dr. Rothe's useful manual contains carefully selected examples on functions of several variables, applications of the calculus to plane curves, and geometrical ideas and transformations associated with complex numbers. The high standard of previous parts is well maintained. T. A. A. B.

BUREAU FOR THE SOLUTION OF PROBLEMS.

This is under the direction of Mr. A. S. Gosset Tanner, M.A., Derby School, Derby, to whom all inquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should wherever possible state the source of their problems and the names and authors of the text-books on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. The names of those sending the questions will not be published.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

INDEX

TO THE

MATHEMATICAL GAZETTE

VOL. XVII.

FEBRUARY 1933, TO DECEMBER 1933

(NOS. 222–226)

TWO SHILLINGS AND SIXPENCE NET

LONDON

G. BELL AND SONS, LTD.

1933

INDEX

TO THE

MATHEMATICAL GAZETTE

INDEX

TO THE

MATHEMATICAL GAZETTE

No. 222, FEBRUARY 1933—No. 226, DECEMBER 1933.

COMPILED BY MRS. T. A. A. BROADBENT.

1. Articles.
2. Mathematical Notes.
3. Reviews and Notices.
4. News of Branches.
5. The Pillory.
6. Gleanings Far and Near.
7. The Library.
8. Obituary Notices.
9. Miscellaneous.

ARTICLES, ETC.

AUTHOR.	TITLE.	PAGE.
S. Barnard.	A proof of the binomial theorem.	294
H. Berry.	The accuracy of figures. Further remarks.	112
F. C. Boon.	Sidetracks in elementary mathematics.	18
L. S. Bosanquet.	The absolute summability of Fourier series.	300
J. P. Dalton.	On the graphical discrimination of the cubic and of the quartic.	189
W. J. Dobbs.	The condition for the equality of the bisectors of two angles of a triangle.	243
N. R. C. Dockeray.	An extension of Van Der Mond's theorem and some applications.	26
M. F. Egan.	On Stirling's theorem as a definition of the Gamma function.	114
D. B. Eperson.	Lewis Carroll—Mathematician.	92
H. Freeman.	The use of symbols of operation in the summation of algebraic series.	177
R. Garver.	The transformation $y=f'(x)$.	251
G. H. Grattan-Guinness.	The question of the momentum.	298
H. G. Green.	Some equations connected with the plane section of the conicoid.	146

Author.	Title.	Page.
T. W. Hall and A. C. Williams.	The graphical equivalent of the principle of moments.	246
W. V. D. Hodge.	Abelian integrals attached to algebraic varieties.	303
A. Lodge.	The envelope of the Simson lines of a triangle.	291
W. H. McCrea.	Operational proofs of some identities.	43
R. F. Muirhead.	Approximating to a square root.	84
V. V. Narlikar.	Dynamics without differential equations.	184
V. Naylor.	Note on the nature of trigonometry.	316
E. H. Neville.	The teaching of geometry.	307
H. T. H. Piaggio.	Mathematics and psychology.	36
D. K. Picken.	The problem of the constant coefficient linear differential operators.	257
	"Products" of vectors.	261
H. E. Piggott.	On relative velocity.	65
	Relative velocity.	263
A. Robson.	Parametric equations in elementary analytical geometry.	87
F. Sandon.	Discussion on the study of statistics in a school course	158
	Discussed by: C. O. Tuckey (p. 162); J. Wishart (p. 164); R. M. Wright (p. 167); H. R. Hamley (p. 173); W. F. Sheppard (p. 175); F. Sandon (p. 175).	
S. T. Shovelton.	Games of Chance. I. The Banker's Clock.	296
A. W. Siddons.	Methods of learning geometrical theorems.	72
	Discussed by: W. Hope-Jones (p. 77); E. F. Edwards (p. 78); F. C. Boon (p. 78); W. C. Fletcher (p. 79); M. P. Meshenberg (p. 79); J. Katz (p. 80); F. A. Roebuck (p. 81); R. M. Wright (p. 81); S. Lister (p. 81); N. F. Sheppard (p. 82); A. W. Siddons (p. 82).	
D. M. Y. Sommerville.	Transportability of a sundial.	46
G. Thomsen.	The treatment of elementary geometry by a group calculus.	230
J. Travers.	Problems connected with a regular polygon of N sides.	313
C. O. Tuckey.	Angles of pedal triangles.	48
	On relative velocity.	70
	Relative velocity.	264
F. Underwood.	The methods of integration of the differential equation $P\,dx+Q\,dy+R\,dz=0$.	105
D. R. Ward.	A new attempt to prove the parallel postulate.	101
G. N. Watson.	Presidential Address, 1933. The Marquis and the Land-agent; a tale of the eighteenth century.	5
J. H. C. Whitehead.	Mathematics at the British Association, 1933.	289
G. J. Whitrow.	Continuity and irrational number.	151
A. C. Williams.	*See* T. W. Hall.	
Annual Meeting of the Mathematical Association, 1933.		1
Report of the Council for the year 1932.		2

MATHEMATICAL NOTES.

Author.	No.	Title.	Page.
R. J. A. Barnard.	1087	Differential equations.	271
F. C. Boon.	1063	Prime numbers.	51
	1070	Morley's triangle.	126
	1085	The converse of Pythagoras' theorem.	270
F. E. Cave.	1064	On Mr. R. J. A. Barnard's article (XVI. pp. 254-5).	51
	1065	On Note 1046.	52
C. I. Davidson.	1080	A method of integration.	206
M. F. Egan.	1096	Stirling's theorem and the Gamma function. Continuity.	328
P. J. Harris.	1086	An approximate construction for "squaring the circle".	271
A. Inglis.	1094	An approximation for $\frac{1}{2}\log\{(1+x)/(1-x)\}$.	327
	1095	The arc of an ellipse.	327
S. Inman.	1093	Further remarks on approximations.	326
B. E. Lawrence.	1083	A note on some circles connected with a triangle.	267
H. V. Mallison.	1073	Euler's theorem.	127
	1084	An extension of Morley's theorem.	268
E. H. Neville.	1075	Continued fractions and series.	200
	1076	A bibliographical note.	200
	1088	A familiar equation.	319
	1089	The vector triple product.	320
	1090	The central point and parameter of the generator of a ruled surface.	321
C. G. Paradine.	1092	Graphical representation of kinetic energy.	323
A. S. Percival.	1074	Note on approximations.	128
A. Robson.		*See* A. W. Siddons.	
J. W. Rogers.	1091	Geometrical proofs of some trigonometrical formulae.	323
A. W. Siddons and A. Robson.	1081	A trigonometrical equation.	266
E. H. Smart.	1078	A problem in maxima and minima and a statical application.	203
C. O. Tuckey.	1077	A "harmonic" set of circles.	201
G. W. Ward.	1066	Successive approximations to $\sqrt{a}$.	52
	1072	Successive approximations to $\sqrt{a}$.	127
J. T. Williams.	1068	Timber lengths.	122
G. Wotherspoon.	1062	To inscribe a regular heptagon in a circle, so that the error for the side shall be less than one-millionth of the radius.	50
C. E. Youngman.	1082	The regular heptagon.	266
Anonymous.	1071	A query concerning a property of Simson's line.	127
	1079	The extended Leibnitz formula.	205
Editorial.	1067	On Notes 1042, 1059.	53
	1069	A difficult converse.	122

REVIEWS AND NOTICES.

Author.	Book.	Reviewer.	Page.
P. Alexandroff.	Einfachste Grundbegriffe der Topologie.	*W. V. D. Hodge.*	54
F. Auerbach.	Lebendige Mathematik.	*E. H. Lockwood.*	345
W. M. Baker.	Analytical geometry of the straight line.	*N. M. Gibbins.*	344
P. B. Ballard.	The child's first number book. The child's second number book. The child's third number book. Teacher's book.	*C. E. O. Body.*	286
H. A. Baxter.	Elementary mechanics of solids.	*R. O. Street.*	346
A. F. Bentley.	Linguistic analysis of mathematics.	*H. G. Forder.*	133
W. G. Bickley.	*See* G. Temple.		
L. Bieberbach.	Differentialgeometrie.	*E. H. Neville.*	56
	Analytische Geometrie. (2).	*E. H. Neville.*	57
	Einleitung in die höhere Geometrie.	*E. H. Neville.*	280
H. Bohr.	Fastperiodische Funktionen.	*N. Wiener.*	54
M. E. Boole.	Collected works. I-IV.	*D. M. Wrinch.*	331
G. Bouligand.	Introduction à la Géométrie infinitésimal directe.	*E. H. Neville.*	342
W. Breidenbach.	Die Dreiteilung des Winkels.	*T. A. A. B.*	288
W. L. Brown.	Related mathematics.	*T. A. A. B.*	64
H. W. Bruns.	Ueben den Ursprung der Tatsache die dem grossen Fermat'schen Theorem zugrunde liegt.	*G. K. Stanley.*	286
G. Cantor.	Gesammelte Abhandlungen.	*D. M. Wrinch.*	332
C. Carathéodory.	Conformal representation.	*P. Fraser.*	130
J. Chazy.	Cours de Mécanique Rationnelle. II. Dynamique des systèmes matériels.	*R. O. Street.*	347
J. M. Child.	Elements of coordinate geometry.	*A. Robson.*	343
S. Cohn-Vossen.	*See* D. Hilbert.		
J. A. Crabb.	Revision exercises in mathematics for general school examinations.	*W. J. Dobbs.*	61
L. Crosland.	School mathematics.	*W. J. Dobbs.*	62
W. F. Crowder.	*See* G. R. Davies.		
C. T. Daltry.	*See* A. W. Siddons.		
G. R. Davies and W. F. Crowder.	Methods of statistical analysis in the social sciences.	*J. Wishart.*	341
L. E. Dickson.	Introduction to the theory of numbers.	*W. E. H. Berwick.*	335
H. Dörrie.	Triumph der Mathematik.	*T. A. A. B.*	143
C. V. Durell.	Matriculation trigonometry.	*H. E. Piggott.*	136
	Advanced algebra. I.	*N. R. C. Dockeray.*	138
G. Dussler.	Spiel und Spielzeug im Physikunterricht.	*L. G. Vedy.*	351
Sir A. S. Eddington.	The expanding universe.	*W. H. McCrea.*	217
S. N. Forrest.	A first trigonometry.	*L. Ashcroft.*	220
J. Frenkel.	Wave mechanics: elementary theory.	*N. F. Mott.*	135

Author.	Book.	Reviewer.	Page.
H. Galbrun.	Théorie mathématique de l'Assurance invalidité et de l'Assurance nuptialité. Définitions et relations fondementales.	*W. Stott.*	221
	Théorie mathématique de l'Assurance invalidité et de l'Assurance nuptialité. Calcul des primes et des réserves.	*W. Stott.*	335
C. M. Gewertz.	Network synthesis.	*S. W. Redfearn.*	338
A. Gminder.	Ebene Geometrie.	*V. Naylor.*	218
W. C. Graustein.	Introduction to higher geometry.	*P. Du Val.*	213
G. A. Hanby.	Geometrical drawing.	*G. M. Cawley.*	60
K. Hayashi.	Tafeln für die Differenzenrechnung sowie für die Hyperbel-, Besselschen, elliptischen und anderer Funktionen.	*L. J. Comrie.*	283
D. Hilbert und S. Cohn-Vossen.	Anschauliche Geometrie.	*H. W. Turnbull.*	277
H. Hilton.	Plane algebraic curves. (2).	*F. Bath.*	212
G. Hoheisel.	Aufgabensammlung zu den gewöhnlichen und partiellen Differentialgleichungen.	*T. A. A. B.*	142
E. L. Ince.	Principles of descriptive geometry.	*M. K. Rice-Ozley.*	220
A. E. Ingham.	The distribution of prime numbers.	*G. Pólya.*	329
H. C. Ives.	Natural trigonometric functions, to seven decimal places, for every ten seconds of arc, together with miscellaneous tables.	*L. J. Comrie.*	284
Sir J. H. Jeans.	The new background of science.	*E. A. Milne.*	274
P. Jordan.	Statistische Mechanik auf quantentheoretische Grundlage.	*W. H. McCrea.*	337
G. Julia.	Exercices d'Analyse. III. Equations différentielles.	*T. A. A. B.*	223
	Essai sur le développement de la théorie des fonctions de variables complexes.	*R. Cooper.*	335
G. Juvet.	Leçons d'Analyse vectorielle. I.	*E. H. Neville.*	280
B. Kerst.	Mathematische Spiele.	*F. C. Boon.*	350
C. J. Keyser.	Humanism and science.	*T. Greenwood.*	134
E. E. Kitchener.	*See* A. S. Pratt.		
C. F. Klein.	Elementary mathematics from an advanced standpoint. Arithmetic. Algebra. Analysis.	*T. A. A. B.*	142
	Vorlesungen über die hypergeometrische Funktion. (2).	*W. N. Bailey.*	333
K. Knopp.	*See* H. von Mangoldt.		
G. Kowalewski.	Interpolation und genäherte Quadratur.	*A. C. Aitken.*	283
Sir H. Lamb.	Hydrodynamics. (6).	*S. Goldstein.*	215
E. P. Lane.	Projective differential geometry of curves and surfaces.	*F. P. White.*	214
D. Larrett.	A senior algebra complete.	*F. C. Boon.*	350

Author.	Title.	Reviewer.	Page.
D. Larrett and J. J. Walton.	Elementary mechanics and hydrostatics.	*J. W. Harmer.*	141
H. Levy.	The universe of science.	*E. H. Neville.*	55
L. Lichtenstein.	Gleichgewichtsfiguren rotierender Flüssigkeiten.	*S. Goldstein.*	341
W. Lietzmann.	Lustiges und Merkwürdiges von Zahlen und Formen. (4).	*T. A. A. B.*	143
	Mathematik und bildende Kunst.	*E. H. Lockwood.*	221
	Funktion und graphische Darstellung.	*W. G. Bickley.*	282
A. Lynch.	The case against Einstein.	*A. Ferguson.*	272
T. M. Macrobert.	Functions of a complex variable. (2).	*T. A. A. B.*	334
D. N. Mallik.	The elements of astronomy. (2).	*R. O. Redman.*	136
H. von Mangoldt und K. Knopp.	Einführung in die höhere Mathematik. II. (6).	*T. A. A. B.*	63
L. M. Mann.	Craftsmen's measures in prehistoric times.	*T. A. A. B.*	64
K. Menger.	Kurventheorie.	*J. H. C. Whitehead.*	220
E. N. Mozley.	Graduated exercises in elementary mathematics.	*W. J. Dobbs.*	62
W. Müller.	Einführung in die Theorie der zähen Flüssigkeiten.	*S. Goldstein.*	215
J. H. Neeley and J. I. Tracey.	Differential and integral calculus.	*F. G. W. Brown.*	141
W. F. Osgood.	Lehrbuch der Funktionentheorie. II. 2.	*W. V. D. Hodge*	276
E. G. Phillips.	An introductory course of mechanics.	*R. O. Street.*	346
H. B. Phillips.	Vector analysis.	*E. H. Neville.*	281
J. Poirée.	La Géométrie à la portée de tous.	*T. M. A. Cooper.*	348
	L'Arithmétique à la portée de tous.	*T. M. A. Cooper.*	348
	L'Algèbre et la Trigonométrie à la portée de tous. I. II.	*T. M. A. Cooper.*	348
J.-B. Pomey.	Application des imaginaires au calcul vectoriel.	*T. A. A. B.*	352
G. Van Praagh.	An introduction to the calculus for science students.	*F. G. W. Brown.*	285
G. Prasad.	Some great mathematicians of the nineteenth century: their lives and their works. I.	*T. A. A. B.*	287
A. S. Pratt and E. E. Kitchener.	Junior primary tests in arithmetic.	*W. J. Dobbs.*	61
E. A. Price.	Logical sequence in plane geometry. Rapid revision by means of ideograms.	*C. E. Kemp.*	349
A. S. Ramsey.	Elementary calculus.	*A. Robson.*	139
	Elementary coordinate geometry.	*A. Robson.*	139
K. Reidemeister.	Knotentheorie.	*H. G. Forder.*	132
J. Rey Pastor.	Teoría de los algoritmos lineales de convergencia y de sumación.	*T. A. A. B.*	286
J. F. Ritt.	Differential equations from the algebraic standpoint.	*E. L. Ince.*	58
F. A. J. Rivett.	A new junior arithmetic.	*W. J. Dobbs.*	61

Author.	Book.	Reviewer.	Page.
R. Rothe.	Höhere Mathematik für Mathematiker, Physiker und Ingenieure. IV. 1.	*T. A. A. B.*	224
	The same; IV. 2.	*T. A. A. B.*	352
H. von Sanden.	Darstellende Geometrie.	*H. G. Forder.*	342
O. Schreier und E. Sperner.	Einführung in die analytische Geometrie und Algebra. I.	*H. G. Forder.*	57
A. W. Siddons and C. T. Daltry.	Elementary algebra. I.	*F. C. Boon.*	349
L. Silberstein.	Causality, a law of nature or a maxim of the naturalist?	*W. H. McCrea.*	338
W. Smith.	Test papers in arithmetic.	*W. J. Dobbs.*	61
R. Smyth.	The quadrilateral: an investigation of its chief properties and a synopsis of old theorems and problems.	*T. A. A. B.*	64
E. Sperner.	*See* O. Schreier.		
M. H. Stone.	Linear transformations in Hilbert space and their applications to analysis.	*J. L. Burchnall.*	131
M. J. O. Strutt.	Lamésche, Mathieusche und verwandte Funktionen in Physik und Technik.	*S. Goldstein.*	59
G. Temple and W. G. Bickley.	Rayleigh's principle and its applications to engineering.	*J. Prescott.*	339
H. Timerding.	Aufgabensammlung zur projektiven Geometrie.	*H. G. Forder.*	343
E. C. Titchmarsh.	The theory of functions.	*A. S. Besicovitch.*	211
J. I. Tracey.	*See* J. H. Neeley.		
C. M. Walsh.	An attempted proof of Fermat's Last Theorem by a new method.	*G. K. Stanley.*	286
J. J. Walton.	*See* D. Larrett.		
N. Wiener.	The Fourier integral and certain of its applications.	*E. C. Titchmarsh.*	129
A. Wisdom.	The Century sum books.	*C. E. O. Body.*	286
R. Wolf.	Geschichte der Astronomie. (Rep.).	*J. K. Fotheringham.*	340

MISCELLANEOUS.

		Page.
Abstracts of dissertations for the degree of Doctor of Philosophy. V.	*T. A. A. B.*	144
Gli Elementi d'Euclide e la critica antica e moderna. Libro X. Edited by F. Enriques.	*Sir T. L. Heath.*	209
The Elements of Euclid. Edited by I. Todhunter.	*T. A. A. B.*	211
Krise und Neuaufbau in den exakten Wissenschaften.	*E. H. Neville.*	330
Mathematical Tables. II. Emden Functions.	*L. M. Milne-Thomson.*	61
The "Omnes" Slide Rule.	*E. J. Atkinson.*	351
University Studies: Cambridge, 1933. Edited by H. Wright.	*E. H. Neville.*	281

BRANCHES.

		PAGE.
London.	Reports of Meetings.	ii, xi
	Programme, 1933-1934.	xix
Queensland.	Report for 1931-1932.	vii
South-West Wales.	Reports of Meetings.	xix
Sydney.	Report for 1932.	xvi
Victoria.	Report for 1932.	xii
Yorkshire.	Reports of Meetings.	iii, vii, xvi

THE PILLORY.

S. Richardson.	London Intermediate B.Sc., 1922.	197

GLEANINGS FAR AND NEAR.

PAGE.	No.	PAGE.	No.	PAGE.	No.
25	898	121	914-915	245	938
35	899	150	916-919	250	939
42	900-902	157	920-921	256	940-941
47	903-904	176	922-924	262	942-943
53	905	183	925-927	265	944-945
71	906	196	928	295	946
83	907	199	929-930	315	947
86	908-910	208	931-932	318	948-949
91	911	229	933-935	328	950
111	912-913	242	936-937		

THE LIBRARY.

DONATIONS.

	PAGE.		PAGE.		PAGE.
J. R. Airey	xvii	G. Garcia	vi, xvii	E. H. Neville	vi, xvii
N. Altshiller-Court	vi	H. N. Haskell	xvii	A. B. Oldfield	vi, xvii
F. C. Boon	xvii	Mrs. E. W. Hobson	xvii	C. E. Phillips	vii
T. A. A. Broadbent	v, xvii	C. E. Kemp	xvii	G. Pólya	vi
J. R. Colthurst	vii	G. Loria	v	G. W. Risdon	xvii
T. M. A. Cooper	xvii	F. L. Marshall	xvii	F. Robbins	xviii
F. W. Dobbs	v	Sir T. Muir	xvii	W. Stott	xviii
H. G. Forder	xvii	V. Naylor	vi	H. E. Tester	xviii
S. Ganguli	vi				

NOTES.

Mr. Dobbs' gift to the Library. xi

The Mathematics Student. xv

Rendiconti del Seminario della Facoltà di Scienze della R.U. di Cagliari. xi

School books, old and new. xv

Scripta Mathematica. xi

OBITUARY.

PAGE.
E. M. Langley. (I. J. P. Kirkman; II. E. T. Bell.) 225

MISCELLANEOUS.

Bureau for the solution of problems. 4, 288, 352
Errata. xviii
Fourth International Congress for Applied Mechanics. 290
Note on the exhibition of books on elliptic functions at the Annual Meeting, 1933. 17
Notice of publication of Index to Vols. I-XV. viii
Notice of proposed Report on the Teaching of Geometry. 104
Royal Statistical Society. 297

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Math. Sem. der Hamburgischen Universität.
American Journal of Mathematics.
American Mathematical Monthly. 39: 9, 10.
Anales de la Sociedad Científica Argentina. 114: 4, 5-6.
Annales de la Société Polonaise de Mathématique.
Annals of Mathematics. 34: 1.
Anuario (Univ. Nac. de la Plata).
Berichte über die Verhandlungen der Akad. der Wiss. zu Leipzig: Math.-Phys. Klasse.
Boletin Matematico. 5: 8, 9.
Boletin Matematico Elemental. 3: 7, 8, 9.
Boletín del Seminario Matemático Argentino. III. 10.
Bollettino della Unione Matematica Italiana. 11: 5.
Bulletin of the American Mathematical Society. 38: 10, 11.
Bulletin of the Calcutta Mathematical Society. 24: 2.
Contribución al Estudio de las Ciencias Fisicas y Matemáticas.
L'Enseignement Mathématique.
Ergebnisse eines Mathematischen Kolloquiums (Wien). 1, 2, 3.
Gazeta Matematica. 38: 3, 4, 5.
Half-Yearly Journal of the Mysore University.
Jahresbericht der Deutschen Mathematiker-Vereinigung.
Japanese Journal of Mathematics.
Journal of the Indian Mathematical Society. 19: 10.
Journal of the London Mathematical Society. 7: 4.
Journal of the Mathematical Association of Japan. 14: 4-5.
Journal de la Société Physico-Mathématique de Léningrade.
Mathematical Notes.
Mathematics Teacher. 25: 7.
Memoria (Univ. Nac. de la Plata).
Monatshefte für Mathematik und Physik.
Nieuw Archief voor Wiskunde. Ser. 2. 17: 3-4.
Periodico di Matematiche.
Proceedings of the Edinburgh Mathematical Society.
Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 14: 9, 10.
Publicaciones . . . Físico-Matemáticas . . . de la Plata.
Publications de la Faculté des Sciences de Masaryk. 153, 154, 156, 158.
Revista de Ciencias (Peru). 397-405.
Revista Matemática Hispano-Americana (Madrid). Ser. 2. 7: 7-8.
Revue Semestrielle des Publications Mathématiques. 37: 5.
School Science and Mathematics. 32: 8, 9.
Science Progress. 107.
Sitzungsberichte der Berliner Mathematischen Gesellschaft. 31.
Studia Mathematica.
Universidad de Zaragoza; Sección de Ciencias. 1: 3.
Unterrichtsblätter für Mathematik und Naturwissenschaften. 38: 9, 10; 39: 1.
Wiskundige Opgaven met de Oplossingen.

BOOKS RECEIVED FOR REVIEW.

A. F. Bentley. *Linguistic Analysis of Mathematics.* Pp. xii, 315. $3.00. 1932. (Principia Press, Bloomington, Indiana)

A. Gminder. *Ebene Geometrie.* Pp. xvi, 490. RM. 22. 1932. (R. Oldenburg, Munich)

A. E. Ingham. *The Distribution of Prime Numbers.* Pp. vii, 114. 7s. 6d. Cambridge Tracts, 30. (Cambridge)

F. Klein. *Elementary Mathematics from an advanced standpoint. Arithmetic—Algebra—Analysis.* Translated from the third German edition by E. R. Hedrick and C. A. Noble. Pp. ix, 274. 15s. 1932. (Macmillan)

J. W. Landon. *Examples in the Theory of Structures.* Pp. viii, 96. 5s. 1932. (Cambridge)

C. I. Lewis and C. H. Langford. *Symbolic Logic.* Pp. xi, 506. $5.00. 1932. (Century Company, New York)

A. Lynch. *The Case against Einstein.* Pp. xxx, 275. 10s. 6d. 1932. (Philip Allan)

W. Müller. *Einführung in die Theorie der zähen Flüssigkeiten.* Pp. x, 367. Geh. RM. 21. Geb. RM. 22.80. 1932. Mathematik in Monographien und Lehrbüchern, 10. (Akademische Verlagsgesellschaft, Leipzig)

J. H. Neeley and J. I. Tracey. *Differential and Integral Calculus.* Pp. viii, 496. 20s. 1932. (Macmillan, New York)

A. S. Ramsey. *Elementary coordinate geometry.* Pp. vii, 155. 4s. 6d. 1932. (Cambridge)

A. S. Ramsey. *Elementary calculus.* Pp. ix, 178. 5s. 1932. (Cambridge)

W. J. Walker. *Numerical Trigonometry and Mensuration.* Pp. 108, 14, xii. With or without answers, 3s. 1932. (Victor White)

C. M. Walsh. *An attempted proof of Fermat's Last Theorem by a new method.* Pp. 41. $1.00. 1932. (G. E. Stechert, New York)

Abstracts of Dissertations for the degree of Doctor of Philosophy. V. Prepared by the Committee for Advanced Studies, University of Oxford. Pp. iv. 121. 3s. 1932. (Oxford; at the Clarendon Press)

Correspondence and Papers of Edmond Halley. Arranged and edited by E. F. MacPike. Pp. xiv, 300. 21s. 1932. (Oxford University Press)

Mathematical Tables. Volume II. Emden Functions, being solutions of Emden's Equation together with certain Associated Functions. Pp. viii, 34. 7s. 6d. 1932. Prepared by the Commission for the Constitution of the Stars of the International Astronomical Union and the British Association Committee for the Calculation of Mathematical Tables. (London, Office of the British Association, Burlington House)

LONDON BRANCH.

On 5th November Mr. Hope-Jones of Eton gave some diverting illustrations of the use of probability in a paper entitled "Probability for School Certificate Candidates". He showed that in logarithmic work mixed addition and subtraction should be far easier than simple addition, since self-contained columns of figures were likely to occur four times as often. If the three angles of a triangle were measured correct to the nearest minute, the probability that their sum would be 180° could be discussed by a graphical process aptly described as "shutting up the truth in a box". This process was applied to interpolation in trigonometric tables. Other illustrations were the determination of the probability that two numbers chosen at random contain no common factor, and the frequency of prime numbers. Mr. Hope-Jones mentioned that Gauss had obtained the same empirical formula for the last piece of work, but had given no demonstration of his method of obtaining it. A new form of lottery involving the number of times a coin could be tossed without two consecutive heads appearing was investigated and found to lead to a functional

law which also occurs in the determination of the number of ancestors in n generations of bees. The whole paper was unique in the freshness and vivacity of the illustrations chosen and in the ease with which the mathematical investigations were carried through.

Dr. Cyril Burt, Professor of Psychology in the University of London (University College), gave the Presidential Address at Bedford College on 3rd December on the subject "Intelligence Tests for, and by Mathematicians". There was a large attendance of nearly 120 members and visitors.

Dr. Burt suggested that psychology had gained much of its repute as a science through enlisting the aid of mathematics. The first stage in the history of intelligence-testing was the study of differences in intelligence. This led to the construction of a scale, and of a unit (the mental ratio) of measurement for intelligence. It was found that mental capacity, so determined, increased by nearly equal annual amounts until the age of fifteen. The distribution of intelligence followed the Normal Curve, like the distribution of height or weight or other physical characteristics.

For the measurement of intelligence after the age of fifteen, the mental ratio unit should be replaced by the Standard Deviation. This measure could be applied to any quantity which could be ranked in order, *e.g.* school marks in a particular subject.

There were very few tests for mathematical ability, but Prof. Spearman had shown that there were probably two specific factors, one for arithmetic and one for mathematics, underlying ability for the subject.

Dr. Burt mentioned that good intelligence tests would correlate as much as ·7, that lists prepared by experienced teachers would correlate as much as ·8, which was much higher than the correlation shown by university scholarship or degree lists.

Dr. Burt concluded with a plea for teaching in secondary schools the rudiments of the mathematical ideas required in the study of intelligence testing, *i.e.* the ideas of frequency distribution, correlation and standard deviation.

C. T. Daltry, Hon. Sec.

YORKSHIRE BRANCH.

The autumn meeting of the Branch was held at the University of Leeds on 12th November. The President, Mr. C. W. Gilham, was in the chair, and there were forty-five members and friends present.

After the transaction of business, Mr. R. M. Gabriel (University of Leeds) gave a most interesting talk on "Cardinals". This was followed by a paper on "The presentation of mathematics to children from 10 to 14" by Colonel Mozley, R.E., which provoked much discussion.

The following committee was elected for the year 1932-1933:

President:	C. W. Gilham (University of Leeds).
Vice-Presidents:	Mrs. Brown (Leeds Training College); W. Todd (Mirfield Grammar School).
Treasurer:	Miss Mathews (Far-Headingley, Leeds).
Secretary:	J. D. Edington (York Training College).
Committee:	J. Anderson (Doncaster); W. Peaker (Leeds); Miss Reeve (Leeds); Miss Brisleden (Wakefield); R. M. Gabriel (Leeds); W. A. Cooper (Ackworth); Miss Hooke (Bradford); H. C. Barnes (Wakefield); A. Montagnon (Leeds).

J. D. Edington, Hon. Sec.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto."—BACON (Preface, *Maxims of the Law*).

President:
Professor G. N. WATSON, Sc.D., F.R.S.

Hon. Treasurer:
F. W. HILL, M.A., 107 Enys Road, Eastbourne, Sussex.

Hon. Secretaries:
C. PENDLEBURY, M.A., 39 Burlington Road, Chiswick, W. 4.
Miss M. PUNNETT, B.A., The London Day Training College, Southampton Row, W.C. 1.

Hon. Librarian:
Professor E. H. NEVILLE, M.A., B.Sc., 160 Castle Hill, Reading, Berks.

Editor of *The Mathematical Gazette*:
T. A. A. BROADBENT, M.A., 2 Buxton Avenue, Caversham, Reading, Berks.

Hon. Secretary of the General Teaching Committee:
ALAN ROBSON, M.A., Marlborough College, Wilts.

Hon. Secretary of the Problems Bureau:
A. S. GOSSET TANNER, M.A., Derby School, Derby.

Hon. Secretary of the Examinations Sub-Committee:
W. J. DOBBS, M.A., 12 Colinette Road, Putney, S.W. 15.

Hon. Secretaries of the Branches:

LONDON:	Miss F. A. YELDHAM, B.Sc., Tralee, Wilmot Way, Banstead, Surrey.
	C. T. DALTRY, B.Sc., 112 Canberra Road, Charlton Park, S.E. 7.
NORTH WALES:	Professor W. E. H. BERWICK, Sc.D., Caerwen, Upper Bangor, North Wales (*pro tem.*).
YORKSHIRE:	A. MONTAGNON, M.A., The Grammar School, Leeds, Yorks.
BRISTOL:	G. W. HINTON, M.A., 32 Tyndalls Park Road, Bristol.
MANCHESTER AND DISTRICT:	A. I. GREGORY, M.A., The County High School for Boys, Altrincham, Cheshire.
CARDIFF:	H. A. HAYDEN, M.Sc., B.Sc., University College, Cardiff.
MIDLAND:	Miss E. M. READ, B.Sc., King's Norton Secondary School.
	Captain A. JACKSON, M.A., King Edward's School, Birmingham.
NORTH EASTERN:	Miss M. WAITE, M.A., The High School, Darlington.
	J. W. BROOKS, B.Sc., 6 Fairholme Avenue, Harton, South Shields, Co. Durham.
LIVERPOOL:	R. BALDWIN, M.A., M.Sc., Wallasey Grammar School, Wallasey, Cheshire.
SYDNEY, N.S.W.:	Miss E. A. WEST, The Girls' High School, North Sydney.
	H. J. MELDRUM, B.A., B.Sc., The Teachers' College, Sydney.
QUEENSLAND:	J. P. MCCARTHY, M.A., Boys' Grammar School, Gregory Terrace, Brisbane.
VICTORIA:	R. J. A. BARNARD, M.A., 21 Bambra Road, Caulfield.
	J. L. GRIFFITH, B.A., 1032 Drummond Street, North Carlton.

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

"The Mathematical Gazette" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least five times a year. The price per copy (to non-members) is usually 3*s.* each.

The *Gazette* contains—Articles, Notes, Reviews, etc., dealing with elementary mathematics, and with mathematical topics of general interest.

THE LIBRARY.

160 Castle Hill, Reading.

The Librarian reports gifts as follows:

From Mr. **T. A. A. Broadbent**, a pamphlet by R. Smyth, and the following:

R. C. Archibald — *Outline of the history of mathematics* - - - 1932
Report of a set of lectures to a Summer School.
Inserted is a signed letter acknowledging a correction.

E. T. Bell — The Queen of the Sciences - - - - - 1931
A volume in the series *A Century of Progress*.

M. E. J. Gheury de Bray — Elementary Hyperbolics (2 vols.) - - - - 1931

C. F. Klein — Elementary Mathematics from an Advanced Standpoint. Arithmetic, Algebra, Analysis - - 1932
Translated from German by E. R. Hedrick and C. A. Noble.

From Mr. **F. W. Dobbs**:

G. B. Airy — Mathematical Tracts (2) - - - - - - 1831

H. Coddington — Optics (2 vols.) - - - - - - - 1829, 1830

J. G. Garnier — Calcul Intégral (3) - - - - - - - - 1812

H. Kuhff — Finite Differences - - - - - - - - 1831

C. F. A. Leroy — Analyse appliquée à la Géométrie des Trois Dimensions - - - - - - - - - - 1829

W. Maddy — Astronomy (; J. Hymers) - - - - - - 1832

J. Narrien — Historical Account of Astronomy - - - - - 1833

G. Peacock — Algebra - - - - - - - - - - 1830
The *Treatise* of 1842-5 was begun as a revised edition of this, but grew to a form to which this description could not be applied.

L. Poinsot — Elémens de Statique (4) - - - - - - 1824

The same (5) - - - - - - - - - 1830

Théorie nouvelle de la Rotation des Corps - - 1834

M. Somerville — Mechanism of the Heavens - - - - - - 1831
Written in the hope of conveying something of the spirit of the *Mécanique Céleste*.

R. Stevenson — Algebraic Equations (2) - - - - - - 1835

Leçons sur le Calcul des Fonctions (2) - - - - - - - 1806
By Lagrange. There was no deliberate anonymity, for the original issue, 1804, had the author's name on the title-page, and the Preface describes the work as a sequel to the *Théorie des Fonctions Analytiques*, a reference which in France at that date could be to only one treatise. A Supplement, 1808, contained the two leçons added in the second edition, reprinted in the quarto form of the first edition. Similar consideration for purchasers of an earlier edition was shown by Dodgson when the second edition of *Euclid and his Modern Rivals* was published in 1885, but it is all too rare.

Syllabus of Differential and Integral Calculus; I (2) - - - - - 1826

The same; II - - - - - - - - - - - - - - 1828
By J. P. Higman. Elementary parts of both differential and integral calculus are treated in I, and II contains "the remainder of the differential calculus".

Also text-books by M. Bland, J. Bonnycastle, P. L. M. Bourdon, H. B. Goodwin, W. Hopkins, J. Hymers (2), D. Lardner, S. D. Poisson, J. C. Snowball, W. Whewell (3), J. Wood, and R. Woodhouse.

From Prof. **G. Loria**, the report of a course of lectures:

Le matematiche nei quaranta secoli della loro storia - - - - 1932

From Prof. **E. H. Neville**:

L. Bieberbach	Analytische Geometrie - - Teubners Leitfäden 29		1930
G. Cassaza	*Quelqu'une des cent manières par les quelles on démontre le cinquème postulat d'Euclide.* A good specimen of its type.		
The British Plutarch	(2 : F. Wrangham) ; V-VI - - - - - -		1816
	First compiled in 1762 by T. Mortimer. Includes lives of Wren, Newton, and Berkeley. *Has any member other odd volumes?*		

From Mr. **A. B. Oldfield**, the numbers wanted to complete the set of *Cambridge Tracts* as far as No. 29 :

W. E. H. Berwick	Integral Bases - - - - - - -	22	1927
C. Carathéodory	Conformal Representation - - - - -	28	1932
L. E. Dickson	Linear Algebras - - - - - - -	16	1914
R. H. Fowler	Differential Geometry of Plane Curves - -	20	1920
G. H. Hardy and M. Riesz	Dirichlet's Series - - - - - - -	18	1915
H. Jeffreys	Operational Methods in Mathematical Physics	23	1927
D. E. Rutherford	Modular Invariants - - - - - -	27	1932
G. C. Steward	Symmetrical Optical System - - - - -	25	1928
E. C. Titchmarsh	Zeta-Function of Riemann - - - - -	26	1930
O. Veblen	Invariants of Quadratic Differential Forms (1 (1927) rep.)	24	1933
	Not a revised edition of J. E. Wright's Tract, No. 9, with this title, but an entirely new tract.		
O. Veblen and J. H. C. Whitehead	Foundations of Differential Geometry - -	29	1932
G. N. Watson	Complex Integration and Cauchy's Theorem	15	1914
L. C. Young	Theory of Integration - - - - - -	21	1927
W. H. Young	Fundamental Theorems of Differential Calculus	11	1910

Also editions fresh to the Library of Nos. 12 (Hardy), 1 (Leathem), 7 (Whittaker). Also a number of the Merriman-Woodward Monographs :

W. E. Byerly	Harmonic Functions (4) - - - - -	5	1906
E. W. Hyde	Grassmann's Space Analysis (4) - - - -	6	1906
W. W. Johnson	Differential Equations (4) - - - - -	9	1906
A. Macfarlane	Vector Analysis and Quaternions (4) - -	8	1906
M. Merriman	Solution of Equations (4) - - - - -	10	1906
L. G. Weld	Determinants (4) - - - - - - -	3	1906
R. S. Woodward	Probability and Theory of Errors (4) - -	7	1906

The third edition, 1900, of a volume of independent chapters, issued in 1896 under the title *Higher Mathematics*, being exhausted, the editors, M. Merriman and R. S. Woodward, decided to issue the eleven chapters as separate monographs, and planned to extend the series ; the monographs which had composed the parent volume were described immediately as in their fourth edition. Ultimately the collection numbered 21, and of these the Library now has all except Nos. 2 (Halsted), 15 (Henderson), 21 (Williams).

From Mr. **N. Altshiller-Court**, offprints of four geometrical papers ; from Prof. **S. Ganguli**, offprints of two papers on the history of mathematics in India ; from Prof. **G. Garcia**, offprint of a paper on classical mechanics and the critical problems of relativity ; from Mr. **V. Naylor**, a school-book by H. Abson ; from Prof. **G. Pólya**, offprints of five papers.

From the Facultad des Ciencias, Buenos Aires :

E. Butty	Física Matemática ; I - - - - - - -		1931

From the Facultés des Sciences, Charles et Masaryk:

V. VOLTERRA *Calcul des Variations* - - - - - - - 1932
Report of two lectures, one on the history of the subject, the other on its place in mathematical physics.

From Rev. **J. R. Colthurst**, a number of publications of the M.A., and from Miss **C. E. Phillips**, a collection of back numbers of the *Gazette*.

The following has been bought:

P. M. H. LAURENT Traité d'Analyse; V, VI, VII - - 1890, 1890, 1891
Completing the set for the Library.

YORKSHIRE BRANCH.

THE spring meeting of the Branch was held on 11th February, 1933, at the Mathematics Department, University of Leeds. The President, Mr. Gilham, was in the chair; there were forty-nine members present.

Professor Brodetsky gave the first paper entitled "The mechanics of the 'Yo-Yo'". Practical demonstrations were given by Mr. Gotcliffe. A discussion on "Trigonometry in schools" was opened by Miss Mathews and Mr. Anderson. The meeting showed that the cutting down of the number of trigonometry questions in the School Certificate papers of the Northern Universities Joint Board was not being received very favourably, but it was decided to see the effect of two sets of papers before discussing the matter in more detail.

J. D. EDINGTON, Hon. Sec.

QUEENSLAND BRANCH.

Report for 1931-1932, presented to the Annual Meeting, 8th April, 1932.

It is with great regret that I make mention of the death of Professor Priestley, to whose efforts the formation of the Branch was due and to whose zeal and enthusiasm it is so much indebted. The sincere sympathy of the Branch is extended to his wife and family.

The last Annual Meeting was held on 27th March, 1931, at the University. The annual report and balance sheet were adopted and officers for the ensuing year were elected. Dr. Simonds addressed the meeting on the subject "Some features of American mathematical teaching".

During the year the usual three meetings have been held. At the first, on 22nd May, 1931, Miss E. H. Raybould read a paper entitled "A Budget of Paradoxes". At the second, on 7th August, Mr. E. W. Jones read a paper on "Entropy", and at the third, on 30th October, Mr. J. C. Deeney read a paper on "A few facts and figures in the Universe".

The number of members of the Branch is thirty, of whom nine are members of the Mathematical Association. Copies of the *Mathematical Gazette* come to hand regularly and are circulated among associate members. The statement of receipts and expenses shows a healthy financial condition, the balance in hand being £6 11s. 4d., which is almost the same as at the end of the previous year. Attendance at meetings during the year has been satisfactory, and I should like to thank those members who have, often at inconvenience to themselves, provided papers for the meeting. J. P. MCCARTHY, Hon. Sec.

FOR SALE. *Mathematical Gazette*, 66-197. Some parts missing. For particulars apply to Major H. A. Harman, Westover, Waterlow Road, Reigate, Surrey.

FOR SALE. *Mathematical Gazette*, 1-204; (April 1894-Dec. 1929). 1-6 bound in quarto, 7-155 in ten octavo volumes, 156-204 (with indexes) unbound. A few of the earlier numbers slightly mildewed, otherwise in practically new condition. £10. Apply: Dr. E. Fenwick, Burley Court, Bath Road, Bournemouth.

BOOKS RECEIVED FOR REVIEW.

H. Abson. *Algebra for Schools.* Pp. viii, 397. 4s. 6d. With answers, 5s. In parts: Part I, 1s. 6d.; Part II, 1s. 9d.; Part III, 2s. Answers to the three parts, 9d. 1933. (University Tutorial Press)

F. Auerbach. *Lebendige Mathematik.* Pp. 355. Geh. RM. 7. Geb. RM. 9. 1929. (Ferdinand Hirt, Breslau)

V. Bjerknes. *C. A. Bjerknes. Sein Leben und seine Arbeit.* Pp. v, 218. Geh. RM. 8.60. Geb. RM. 9.80. 1933. (Springer, Berlin)

G. Bouligand. *Introduction à la géométrie infinitésimale directe.* Pp. viii, 229. 36 fr. 1932. (Vuibert, Paris)

H. W. Bruns. *Ueben den Ursprung der Tatsache die dem grossen Fermat'schen Theorem zugrunde liegt.* Pp. 15. No price stated. 1933. (B. Wepf, Basel)

N. J. Chignell and **E. H. Fryer.** *An Introduction to Co-ordinate Geometry and the Calculus.* Pp. vii, 231. 5s. 1933. (Blackie)

H. Dörrie. *Triumph der Mathematik.* Pp. vii, 386. Geh. RM. 7. Geb. RM. 9. 1933. (Ferdinand Hirt, Breslau)

G. Dussler. *Spiel und Spielzeug im Physikunterricht.* Pp. 79. RM. 2.24. 1932. Beihefte der Unterrichtsblätter für Mathematik und Naturwissenschaften, 13. (Otto Salle, Frankfurt a. M.)

Sir Arthur Eddington. *The Expanding Universe.* Pp. vii, 128. 3s. 6d. 1933. (Cambridge)

F. Enriques. *Gli elementi d'Euclide e la critica antica e moderna:* III. *Libro X.* Pp. 333. L. 30. 1932. Per la storia e la filosofia delle matematiche, 10. (Zanichelli, Bologna)

S. N. Forrest. *A First Trigonometry.* Pp. 96. 1s. 9d. 1933. (Arnold)

H. Hancock. *Foundations of the Theory of Algebraic Numbers.* II. Pp. xxvi, 654. $8.00. 1932. (The Macmillan Company, New York)

K. Hayashi. *Tafeln für die Differenzenrechnung sowie für die Hyperbel-, Besselschen, elliptischen und anderen Funktionen.* Pp. vi, 66. RM. 12. 1933. (Springer, Berlin)

G. Hoheisel. *Aufgabensammlung zu den gewöhnlichen und partiellen Differentialgleichungen.* Pp. 148. RM. 1.62. 1933. Sammlung Göschen, 1059. (Walter de Gruyter, Berlin)

E. L. Ince. *Principles of Descriptive Geometry.* Pp. viii, 152. 8s. 6d. 1933. (Arnold)

G. Juvet. *Leçons d'Analyse vectorielle. I. Géométrie différentielle des courbes et des surfaces. Théorie mathématique des champs.* Pp. 120. Francs suisses 8. 1933. (Rouge, Lausanne; Gauthier-Villars, Paris)

W. Lietzmann. *Funktion und graphische Darstellung.* Pp. 190. Geh. RM. 4.50. Geb. RM. 6. 1925. (Ferdinand Hirt, Breslau)

W. Lietzmann. *Lustiges und Merkwürdiges von Zahlen und Formen.* 4th edition. Pp. vi, 307. Geb. RM. 8.50. 1930. (Ferdinand Hirt, Breslau)

J. Rey Pastor. *Teoría de los algoritmos lineales de convergencia y de sumación.* Pp. 174. No price stated. 1931. (Imprenta de la Universidad, Buenos Aires)

N. Wiener. *The Fourier Integral and certain of its Applications.* Pp. xi, 201. 12s. 6d. 1933. (Cambridge)

INDEX TO VOLUMES I-XV.

With this number of the *Gazette* is issued the index to Volumes I-XV to which reference has more than once been made. Although complete sets of these volumes are rare, most of the numbers of Volumes XI-XV are still in print with the publishers and may be ordered from any bookseller. The Association holds a miscellaneous stock of the earlier numbers and the editor will supply members whose curiosity has been provoked with these numbers, as far as possible, at the cost of 1s. each.

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Math. Sem. der Hamburgischen Universität. 9: 2.
American Journal of Mathematics. 55: 1, 2.
American Mathematical Monthly. 40: 1, 2, 3, 4, 5.
Anales de la Sociedad Científica Argentina. 115: 1, 2, 3, 4.
Annales de la Société Polonaise de Mathématique. 9.
Annali della R. Scuola . . . di Pisa. Ser. 2. 2: 1, 2.
Annals of Mathematics. Ser. 2. 34, 2.
Anuario (Univ. Nac. de la Plata).
Berichte über die Verhandlungen der Akad. der Wiss. zu Leipzig: Math.-Phys. Klasse. 83; 2, 3, 4, 5, 6; 84: 1, 2, 3, 4.
Boletin Matematico. 5: 10, 11, 12.
Boletin Matematico Elemental. 3: 10, 11, 12; 4: 1. 2.
Boletín del Seminario Matemático Argentino. 3: 11.
Bollettino della Unione Matematica Italiana. 12: 1, 2.
Bulletin of the American Mathematical Society. 38: 12; 39: 1, 2, 3, 4.
Bulletin of the Calcutta Mathematical Society. 24: 3.
Contribución al Estudio de las Ciencias Fisicas y Matemáticas.
L'Enseignement Mathématique. 31: 1-3.
Gazeta Matematica. 38: 6, 7, 8, 9.
Half-Yearly Journal of the Mysore University.
Jahresbericht der Deutschen Mathematiker-Vereinigung. 42: 5-8.
Japanese Journal of Mathematics. 8: 4; 9: 1, 2.
Journal of the Indian Mathematical Society. 19: 11, 12.
Journal of the London Mathematical Society. 8: 1, 2.
Journal of the Mathematical Association of Japan. 14: 6; 15: 1, 2.
Journal de la Société Physico-Mathématique de Léningrade.
Mathematical Notes.
Mathematics Teacher. 25: 8; 26: 1, 2, 3, 4, 5.
Memoria (Univ. Nac. de la Plata).
Monatshefte für Mathematik und Physik.
Periodico di Matematiche. Ser. 4. 12: 5; 13: 1.
Proceedings of the Edinburgh Mathematical Society. Ser. 2. 3: 3.
Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 14: 11; 15: 1, 2, 3.
Publicaciones . . . Fisico-Matemáticas . . . de la Plata.
Rendiconti del Seminario della Fac. di Sci. della R.U. di Cagliari. 1: 2, 3, 4; 2: 1, 2, 3, 4; 3: 1.
Revista de Ciencias (Peru). 397-405.
Revista Matemática Hispano-Americana (Madrid). Ser. 2. 7: 9, 10.
Revue Semestrielle des Publications Mathématiques. 37: 6; 38: 1.
School Science and Mathematics. 33: 1, 2, 3, 4, 5.
Science Progress. 108.
Scripta Mathematica. 1: 1, 2, 3.
Studia Mathematica.
Trabajos del Seminario Matemático [Argentino]. 1932.
Universidad de Zaragoza. Sección de Ciencias. 1: 4.
Unterrichtsblätter für Mathematik und Naturwissenschaften. 39: 2, 3, 4, 5.
Wiskundige Opgaven met de Oplossingen. 15: 4.

BOOKS RECEIVED FOR REVIEW

P. B. Ballard. *The Child's First Number Book. The Child's Second Number Book. The Child's Third Number Book.* Pp. 32 each. Paper 6d., limp cloth 8d. each. *Teacher's Book.* Pp. 47. Limp cloth 2s. 1933. (University of London Press)

E. W. Barnes. *Scientific Theory and Religion.* Pp. xxiv, 685. 25s. 1933. (Cambridge)

L. Bieberbach. *Einleitung in die höhere Geometrie.* Pp. viii, 128. RM. 6.40. 1933. Teubners mathematische Leitfäden, 39. (Teubner)

G. D. Birkhoff. *Aesthetic Measure.* Pp. xv, 226. 42s. 1933. (Harvard University Press; Humphrey Milford)

W. Breidenbach. *Die Dreiteilung des Winkels.* Pp. iv, 38. RM. 1.20. 1933. Mathematisch-Physikalische Bibliothek, Reihe I, Band 78. (Teubner)

E. W. Brown and D. Brouwer. *Tables for the development of the Disturbing Function.* Pp. 69-157. 10s. 6d. 1933. (Cambridge)

J. M. Child. *Elements of coordinate geometry.* Pp. xiii, 468. 12s. 6d. 1933. (Macmillan)

H. Galbrun. *Théorie mathématique de l'Assurance invalidité et de l'Assurance nuptialité. Définitions et relations fondamentales.* Pp. 156. 40 fr. 1933. E. Borel's *Traité du Calcul des Probabilités et de ses Applications*, Tome III, Fasicule 4. (Gauthier-Villars)

C. M. Gewertz. *Network Synthesis.* Pp. vi, 257. 23s. 1933. (Baillière, Tindall and Cox)

F. J. Hemmings and J. P. Chalk. *Trigonometry for schools.* Pp. viii, 266. 4s. 6d. 1933. (Blackie)

G. Julia. *Exercices d'Analyse. III. Équations différentielles.* Pp. iv, 287. 60 fr. 1933. (Gauthier-Villars)

G. Julia. *Essai sur le développement de la théorie des fonctions de variables complexes.* Pp. viii, 53. 12 fr. 1933. (Gauthier-Villars)

G. Kowalewski. *Interpolation und genäherte Quadratur.* Pp. v, 146. RM. 9.60. 1932. (Teubner)

P. Lenard. *Great Men of Science.* Pp. xix, 389. 12s. 6d. 1933. (Bell)

W. Lietzmann. *Mathematik und bildende Kunst.* Pp. 150. Geh. RM. 4. Geb. RM. 5.50. 1931. (Ferdinand Hirt, Breslau)

K. Menger. *Kurventheorie.* Pp. vi, 376. Geh. RM. 22. Geb. RM. 24. 1932. Mengentheoretische Geometrie in Einzeldarstellungen, 2. (Teubner)

K. Ogura. *History of Mathematical Education.* I. Pp. iv, 4, 7, 356. 2.50 yen. 1932. (Iwanami Shoten) [In Japanese]

H. B. Phillips. *Vector Analysis.* Pp. viii, 236. 15s. 6d. 1933. (John Wiley; Chapman and Hall)

G. van Praagh. *An introduction to the calculus.* Pp. ix, 92. 2s. 6d. 1933. (Macmillan)

R. Rothe. *Höhere Mathematik für Mathematiker, Physiker und Ingenieure.* Teil IV, Heft 1. Pp. ii, 52. RM. 2. 1932. Teubners mathematische Leitfäden, 33. (Teubner)

H. Timerding. *Aufgabensammlung zur projektiven Geometrie.* Pp. 140. RM. 1.62. 1933. Sammlung Göschen, 1060. (Walter de Gruyter, Berlin)

O. Veblen. *Projektive Relativitätstheorie.* Pp. v, 73. RM. 8. 1933. Ergebnisse der Mathematik, Band II, Heft 1. (Springer)

A. N. Whitehead. *Adventures of Ideas.* Pp. xii, 392. 12s. 6d. 1933. (Cambridge)

A. Wisdom. *Century sum books.* I A, I B; II A, II B. Pp. 48 each. Paper 7d., limp cloth 8d. each. 1933. (University of London Press)

The Elements of Euclid. Edited by Isaac Todhunter. Introduction by Sir Thomas Heath. Pp. xviii, 298. 2s. Everyman's Library, 891. (Dent)

SCRIPTA MATHEMATICA.

AMONG victims of the war was *Bibliotheca Mathematica*, of which publication was suspended in 1915. A persistent attempt to revive Enestrŏm's journal some years ago failed; there was interest in plenty on both sides of the Atlantic, but in an unstable period promises of definite support were too few. Yeshiva College, New York, has now undertaken, not indeed to replace the *Bibliotheca*, but to give us "a quarterly journal devoted to the philosophy, history, and expository treatment of mathematics"; whatever misgivings the mention of the last object may arouse are dispelled by the names of the editor and his assistants and by the most casual glance through the numbers that have already appeared, no less than by a careful explanation in the note which introduces the first number.

The articles are short, and appeal to so wide a range of human and literary interest that to quote only a few titles would be misleading. Of the books reviewed in the first three numbers, only one is itself purely mathematical, and the review of this one is a historical essay on the subject, not an account or a criticism of the author's contributions. Finally, R. C. Archibald, an old friend of the *Gazette*, is in charge of a section of Notes and Queries; this section should serve one of the purposes for which the defunct *Intermédiaire des Mathématiciens* was formed, and it includes also material for a biographical and bibliographical card-index of surprising range.

The new quarterly fills a vacant place, and should receive a very warm welcome; we wish it every success.

UNIVERSITY OF CAGLIARI.

THE Library is now receiving the *Rendiconti del Seminario della Facoltà di Scienze della R. U. di Cagliari*, a 32-page quarterly begun in 1931. The exchange has been made retrospective, in order that our set may be complete. The mathematical editor is Prof. U. Crudeli, and in the first two volumes mathematics occupies about a quarter of each number. We welcome the evidence of intellectual activity provided by our new contemporary.

MR. DOBBS' GIFT TO THE LIBRARY.

MEMBERS will be interested to know that the books given by Mr. F. W. Dobbs, reported in the May *Gazette*, were used at Cambridge by his grandfather, J. T. Chance, of Trinity, 7th Wrangler in 1838, who was afterwards head of the celebrated glass works associated with his name. Most of the books were bound in ill-assorted twos and threes, and the boards were falling away from the volumes; Mr. Dobbs has added greatly to the value of his gift by having the volumes submitted to a thorough overhaul, in consequence of which forced partnerships are dissolved and a stout buckram replaces the unserviceable smooth calf that was fashionable a century ago.

LONDON BRANCH.

A MEETING was held at Bedford College on 28th January, 1933, for Annual Business and a paper by Mr. J. Katz on "Experimenting with School Certificate work in the first years of algebra". Owing probably to the influenza epidemic only twenty-one members and visitors attended. The Annual Report of the committee showed a slight decrease in membership due to older members dropping out, but mentioned that younger members had joined steadily throughout the year and that the average attendance at meetings continued to increase. Professor S. Brodetsky was elected President of the Branch and Mr. F. C. Boon was elected Chairman.

In his paper Mr. Katz said that he had three ideas in mind when he started his experiment:

(*a*) From the ages of 12 to 15 a boy is at the peak of his biological ability: should it not be possible for him then to assimilate that part of the Certificate course that depended purely on natural intelligence?

(*b*) Modern treatment of algebra leads to weakness of manipulative power. Was it possible to increase this power in some stimulating way?

(*c*) Could not the traditional sequence in algebra be replaced by another with results equally beneficial in the end?

Mr. Katz began with square root, leading to Pythagoras' theorem, and simple binomial expansions, using detached coefficients and Pascal's triangle. In the second term, quadratics by factors, division, infinite geometric progressions and, finally, arithmetic progressions were taken. The third term was used for revision and extension, and the remainder theorem. It was found that such a course was easily assimilated, without causing less manipulative power. Mr. Katz concluded that there was no reason at all for not altering the traditional sequence in algebra.

The usual Members' Topics meeting was held on 18th February and caused unusually vigorous discussion. Fifty-six members and visitors were present. The following topics were introduced:

"Tabulation in mathematics", by Mr. Clark;

"Tips in factorisation", by Mr. Siddons;

"Introducing fractional and negative indices", by Mr. Blamey;

"A new system of decimal coinage", by Mr. Inman.

On 18th March, Mr. Dockeray, of Dulwich College, read a paper on "The Theory of Numbers". Thirty-five members and visitors were present. The paper was divided into three sections. First came the early history of the subject, including the discoveries relating to perfect and amicable numbers, and different types of indeterminate equations. The second section was devoted to the solution of congruences of the first and second degree. Legendre's law of quadratic reciprocity was proved by a geometrical device due to Gauss. Finally, an isolated theorem of Fermat's was considered in some detail.

C. T. DALTRY, Hon. Sec.

VICTORIA BRANCH.

REPORT FOR THE YEAR 1932.

THE office-bearers for the year 1932 were: *Hon. President:* Professor Nanson. *President:* Professor Cherry. *Vice-Presidents:* Dr. Baldwin, Miss Laing, Professor Michell, Mr. Picken, Mr. Seitz. *Committee:* Miss Kirkhope, Mr. Lane, Miss Waddell. *Secretaries:* Mr. Barnard, Mr. Griffiths. *Treasurer:* Mr. Willoughby.

The total number of members for the year 1932 was 18, of whom 8 were members of the parent society.

Four meetings were held during the year, all of which were well attended and aroused much interest and discussion. On 9th May, Mr. E. O. Willoughby gave a paper on "Probability from the time of the French gamblers". The paper dealt with probability mainly as affecting games of chance. On 19th June, Professor Cherry gave a paper on "Navigation of Interplanetary Space". He dealt with the various difficulties and contradictions of physical laws involved in various accounts of voyages to the moon. On 23rd July, Mr. R. J. A. Barnard gave a paper on "Greek Arithmetic", dealing mainly with the methods of numerical approximation used by Archimedes. On 12th September, Miss W. Waddell initiated a discussion on the teaching of geometry for the Intermediate standard. Much of the discussion turned on the question as to how far the fundamental facts, such as those relating to the congruence of triangles, could be assumed without proof in school teaching.

R. J. A. BARNARD.

JOURNALS RECEIVED.

When no number is attached, no part has been received since a previous acknowledgment.

Abhandlungen aus dem Math. Sem. der Hamburgischen Universität.
American Journal of Mathematics. 55: 3.
American Mathematical Monthly. 40: 6.
Anales de la Sociedad Científica Argentina. 115: 5, 6; 116: 1, 2.
Annales de la Société Polonaise de Mathématique.
Annali della R. Scuola di Pisa. Ser. 2. 2: 3.
Annals of Mathematics. Ser. 2. 34, 3.
Berichte über die Verhandlungen der Akad. der Wiss. zu Leipzig: Math.-Phys. Klasse. 84: 5; 85: 1, 2.
Boletin Matematico. 6: 1.
Boletin Matematico Elemental. 4: 3, 4, 5.
Boletín del Seminario Matemático Argentino. 3: 12.
Bollettino della Unione Matematica Italiana. 12: 3.
Bulletin of the American Mathematical Society. 39: 5, 6, 7, 8.
Bulletin of the Calcutta Mathematical Society.
Contribución al Estudio de las Ciencias Físicas y Matemáticas.
L'Enseignement Mathématique. 31: 4-6.
Ergebnisse eines Mathematischen Kolloquiums (Wien). 4.
Gazeta Matematica. 38: 10, 11, 12.
Half-Yearly Journal of the Mysore University.
Jahresbericht der Deutschen Mathematiker-Vereinigung. 42: 9-12; 43: 1-4.
Japanese Journal of Mathematics.
Journal of the Indian Mathematical Society.
Journal of the London Mathematical Society. 8: 3.
Journal of the Mathematical Association of Japan. 15: 3.
Mathematical Notes.
Mathematics Student. 1: 1.
Mathematics Teacher.
Memoria (Univ. Nac. de la Plata).
Monatshefte für Mathematik und Physik.
Nieuw Archief voor Wiskunde.
Periodico di Matematiche.
Proceedings of the Edinburgh Mathematical Society.
Proceedings of the Physico-Mathematical Society of Japan. Ser. 3. 15: 4, 5, 6, 7.
Publicaciones . . . Físico-Matemáticas . . . de la Plata.
Publications de la Faculté des Sciences de Masaryk. 163, 165, 168, 171, 172.
Rendiconti del Seminario della Fac. di Sci. della R.U. di Cagliari. 3.
Revista de Ciencias (Peru). 406-408.
Revista Matemática Hispano-Americana (Madrid).
Revue Semestrielle des Publications Mathématiques. 38: 2, 3.
School Science and Mathematics. 33: 6.
Scripta Mathematica. 1: 4.
Sitzungsberichte der Berliner Mathematischen Gesellschaft.
Universidad de Zaragoza. Sección de Ciencias. 2: 1, 2.
Unterrichtsblätter für Mathematik und Naturwissenschaften. 39: 6, 7.
Wiskundige Opgaven met de Oplossingen. 16: 1.

BOOKS RECEIVED FOR REVIEW.

W. M. Baker. *Analytical geometry of the straight line.* Pp. i, 63, v. Limp cloth, 1s. 6d. 1933. (Bell)

S. Banach. *Théorie des opérations linéaires.* Pp. viii, 254. $3. 1932. Monografje Matematyczne, 1. (Warsaw)

H. A. Baxter. *Elementary mechanics of solids.* Pp. x, 281. 8s. 6d. 1933. (Blackie)

E. T. Bell. *Numerology.* Pp. vii, 187. 11s. 6d. 1933. (Williams and Wilkins, Baltimore; Baillière, Tindall and Cox)

J. Chazy. *Cours de Mécanique Rationnelle. II. Dynamique des systèmes matériels.* Pp. vi, 462. 80 fr. (Gauthier-Villars)

G. R. Davies and W. P. Crowder. *Methods of statistical analysis in the social sciences.* Pp. xi, 355. 20s. 1933. (John Wiley; Chapman and Hall)

S. Dawson. *An introduction to the computation of statistics.* Pp. 192. 10s. 6d. 1933. (University of London Press)

C. V. Durell and A. Robson. *Elementary calculus. I.* Pp. viii, 240. 4s. 6d. Without appendix, 3s. 6d. 1933. (Bell)

P. Franklin. *Differential equations for electrical engineers.* Pp. vii, 299. 16s. 6d. 1933. (John Wiley; Chapman and Hall)

D. Hilbert. *Gesammelte Abhandlungen. II. Algebra. Invariantentheorie. Geometrie.* Pp. viii, 453. RM. 45. 1933. (Springer)

Sir James Jeans. *The new background of science.* Pp. viii, 303. 7s. 6d. 1933. (Cambridge)

P. Jordan. *Statistische Mechanik auf quantentheoretische Grundlage.* Pp. xi, 112. RM. 8.20. 1933. Die Wissenschaft, 87. (Vieweg, Braunschweig)

B. Kerst. *Mathematische Spiele.* Pp. vi, 90. RM. 2. Linen, RM. 3.20. 1933. (Grote, Berlin)

F. Klein. *Vorlesungen über die hypergeometrische Funktion.* Edited by O. Haupt. Pp. ix, 344. Geh. RM. 22. Geb. RM. 23.60. 1933. Grundlehren der math. Wiss., 39. (Springer)

D. Larrett. *A senior algebra complete.* Pp. 308. Without answers, 3s. 6d.; with answers, 4s. 1933. (Harrap)

L. Lichtenstein. *Gleichgewichtsfiguren rotierender Flüssigkeiten.* Pp. viii, 175. RM. 15.60. 1933. (Springer)

T. M. Macrobert. *Functions of a complex variable.* Second edition. Pp. xv, 347. 14s. 1933. (Macmillan)

F. M. Marzials. *The groundwork of geometry, plane and solid.* Pp. xiv, 198. 3s. 6d. Or in three parts, 1s. 6d. each. 1933. (Nelson)

E. G. Phillips. *An introductory course of mechanics.* Pp. viii, 255. 10s. 6d. 1933. (Cambridge)

J. Poirée. *L'Arithmétique à la portée de tous.* Pp. 96. 25 fr. 1932. (Gauthier-Villars)

J. Poirée. *L'Algèbre et la Trigonométrie à la portée de tous.* Tome I. *Calcul algébrique et équations.* Pp. v, 56. 15 fr. Tome II. *Étude de la variation des fonctions.* Pp. v, 43. 15 fr. 1933. (Gauthier-Villars)

J. Poirée. *La Géométrie à la portée de tous.* Pp. 117. 20 fr. 1931. (Gauthier-Villars)

J.-B. Pomey. *Application des imaginaires au calcul vectoriel.* Pp. vi, 70. 20 fr. 1933. (Gauthier-Villars)

G. Prasad. *Some great mathematicians of the nineteenth century. I.* Pp. xv, 347. RM. 6. 1933. (Benares Mathematical Society; K. F. Koehlers Antiquarium, Leipzig)

T. Radó. *On the problem of Plateau.* Pp. 109. RM. 12.80. 1933. Ergebnisse der Mathematik, Band II, Heft 2. (Springer)

S. Saks. *Théorie de l'intégrale.* Pp. viii, 290. $4. 1933. Monografje Matematyczne, 2. (Warsaw)

L. Silberstein. *Causality, a law of nature or a maxim of the naturalist?* Pp. viii, 159. 6s. 6d. 1933. (Macmillan)

W. Threlfall. *Gruppenbilder.* Pp. 59. Geh. RM. 3.30. 1932. (Hirzel)

W. J. Walker. *Higher Certificate Algebra.* Part III of *New School Algebra.* Pp. viii, 259-394, 59-64. With or without answers, 3s. 1933. (Mills and Boon)

R. Wolf. *Geschichte der Astronomie.* Reprint of 1877 edition. Pp. xvi, 815. RM. 15. 1933. (R. Oldenbourg, Munich; K. F. Koehlers Antiquarium, Leipzig)

Contributions to the calculus of variations, 1931-1932. Theses submitted to the department of mathematics of the University of Chicago. Pp. vii, 523. 16s. 6d. 1933. (University of Chicago Press; Cambridge University Press)

Krise und Neuaufbau in den exakten Wissenschaften. Pp. ii, 122. RM. 3.60. 1933. (Deuticke, Leipzig and Vienna)

University Studies: Cambridge 1933. Edited by H. Wright. Pp. ix, 294. 10s. 6d. 1933. (Ivor Nicholson and Watson)

THE MATHEMATICS STUDENT.

Before 1923, the *Journal of the Indian Mathematical Society* completed a volume each year. Then the style of the individual numbers was changed, a section entitled *Notes and Questions*, though not separated from the main journal, being paged independently. In order that a volume, if split into its two components, should not be too small, each volume was designed to run through two years, and in this way vols. 15-19 covered the period 1923-1932. There is now to be complete separation, the more solid part continuing as the *Journal*, and the more elementary contributions forming a quarterly, *The Mathematics Student*, of which the first number has just reached us. When we say that a portrait of Prof. Hobson forms a frontispiece, and that the first article is a fascinating one, "Some glimpses of ancient Indian mathematics" by Mr. Krishnaswami Ayyangar, an old contributor to the *Gazette*, it will be evident that the new periodical will play its part in maintaining the happy relations between English and Indian mathematicians.

SCHOOLBOOKS, OLD AND NEW.

In offering a number of recent schoolbooks to the Library, Mr. T. M. A. Cooper remarks that he is thinking not of their value to a library to-day, but of their importance to the future student of the history of teaching; books that are beneath the notice of the great libraries and of the compilers of catalogues become extremely difficult to find after a decade or so, and the Association is pursuing a sound policy in building up a collection of schoolbooks, irrespective of their merits and of their contemporary success. This is a view with which the Librarian agrees most emphatically, and he takes this opportunity to put it again before members. Every teacher accumulates yards of out-of-date junk on his shelves—or shelves of unsaleable junk in his yard—and the Association is surely a good alternative to the bonfire. Even a book which already appears in our lists may be welcome, since the copy at present possessed may easily be a dilapidated and dog-eared one, while a book of which we have one edition is likely to be exceptionally valuable, since nothing shows the strength of currents of reform more clearly than the changes made in a book that has once established its position.

SYDNEY BRANCH.

Report for the year 1932.

Connected with this branch there are now 18 members and 104 associates. During the year, three meetings were held. At the first of these Mr. Lyons gave a very interesting address on "The Foundations of Geometry". The modern development of geometry, which was independent of the concepts of metrical properties as a basis, was outlined.

The second address was given by Professor C. E. Weatherburn, during his visit to Sydney in connection with the meeting of the A.A.A.S., on "The Tensor Calculus". The recent developments of this branch of mathematics were ably outlined.

At the Annual Meeting the reports of the various office-bearers were received. The election of officers for 1933 was as follows: *President*: Professor H. S. Carslaw; *Hon. Treasurer*: Mr. A. L. Nairn; *Hon. Secretaries*: Miss E. A. West, Mr. H. J. Meldrum. There was a short discussion on the mathematical papers set for the various school examinations for 1932.

An address was given by Mr. Denning on "Applied Mathematics". The speaker said that his aim was to introduce a discussion, pointing out the developments in the modern world of business and economics, in which directions applied mathematics was essential. He suggested that it should be possible to give a greater bias in school mathematics to this applied work. A revision of the mathematical syllabus seemed desirable. Time did not allow of any adequate discussion. It is hoped that the subject will be brought up for discussion at some future meeting.

E. A. West,
H. J. Meldrum, } Hon. Secs.

YORKSHIRE BRANCH.

11th March, 1933. The Annual Dinner was held at The Griffin Hotel, Leeds. The President, Mr. Gilham, was in the chair and there were 32 members and friends present.

The guests were Professor Turnbull of St. Andrews, and Mr. W. F. Beard, an ex-president of the Branch.

The president proposed the toast "Our Guests"; each guest replied, Professor Turnbull giving an interesting account of some research into old documents at St. Andrews.

"Euclid" was proposed by Mr. Liddle and Professor Milne replied. The President was proposed by Miss Brisleden and Mr. Gilham replied.

Dr. Crowe gave a selection of songs and Mr. Montagnon acted as toastmaster.

20th May, 1933. Summer Meeting held at The High School for Girls, Skipton. The President in the chair and 35 members present.

The first paper was given by Dr. Stoneley on "The Discovery of Pluto". Mr. Peaker introduced a discussion on "Accuracy" and many members expressed their opinions.

A very hearty vote of thanks was proposed to Miss Wise and Staff for their hospitality.

J. D. Edington, Hon. Sec.

0.5 Math.

THE LIBRARY OF THE

THE LIBRARY.

160 CASTLE HILL, READING.

JAN 10 1934

THE Librarian reports gifts as follows :

UNIVERSITY OF ILLINOIS

From Dr. **J. R. Airey** :

W. SCHEIBNER — Zur Reduction elliptischer Integrale in reeller Form — 1879

From Mr. **F. C. Boon**, a school book by A. W. Siddons and C. T. Daltry.

From Mr. **T. A. A. Broadbent** :

EUCLID — Elements I-VI, XI, XII - - - - - - - - 1933
Todhunter's revision of Simson's edition, reproduced photographically in *Everyman's Library*, with an introduction by T. L. Heath.

James Clerk Maxwell, 1831-1931 - - - - - - - - - 1931
The ten addresses delivered at the centenary celebrations at Cambridge.

From Mr. **T. M. A. Cooper**, school books by W. S. Beard, H. McKay (3 pairs of volumes), W. Johnson (3 volumes), and W. Smith (5 pairs of volumes).

From Mr. **H. G. Forder** :

H. TIMERDING — Aufgabensammlung zur Projektiven Geometrie - 1933

From Prof. **G. Garcia**, an offprint of a paper on projectiles.

From Mr. **H. N. Haskell**, a pamphlet :

J. J. GLIEBE — *Trisecting an angle* - - - - - - - - 1932
"An age-old problem solved." A sumptuous edition has since appeared under the title *The Mathematical Atom*.

From Mrs. **E. W. Hobson**, three Italian pamphlets needed to render the set of publications of the Commission Internationale de l'Enseignement Mathématique complete.

From Mr. **C. E. Kemp**, a collection of back numbers of the *Gazette*, and a pamphlet :

E. A. PRICE — *Logical sequence in plane geometry : rapid revision by means of ideograms* - - - - - - 1933

From Miss **F. L. Marshall**, school books by W. G. Borchardt and A. D. Perrott (2), A. D. Capel, H. H. Champion and J. A. C. Lane, C. V. Durell and G. W. Palmer, M. Eastwood and J. Lightfoot, W. Farquharson and H. W. Carter, J. W. M. Gunn (2), T. W. Piper, E. Sankey and A. Royds, S. G. Starling and F. C. Clarke, A. Thom, A. E. Tweedy (2), and T. S. Usherwood and C. J. A. Trimble, together with a number of pamphlets.

From Sir **T. Muir**, a number of his most recent papers on determinants.

From Prof. **E. H. Neville**, school books by W. M. Baker and A. A. Bourne, W. G. Borchardt and A. D. Perrott, J. Davidson and A. J. Pressland, R. C. Fawdry, R. W. M. Gibbs, J. Gray and F. J. Smith, A. B. Grieve, P. J. Haler and A. H. Stuart, H. S. Hall and F. H. Stevens, R. Lachlan and W. C. Fletcher, J. G. Leathem, and W. Parkinson and A. J. Pressland, together with

F. F. P. BISACRE — Applied Calculus - - - - - - - - 1921

H. S. MAXIM — Monte Carlo Facts and Fallacies - - - - 1904
A serious discussion of systems and runs.

A. TACQUET — Opera Mathematica - - - - - - - - 1669

From Mr. **A. B. Oldfield** :

A. E. INGHAM — Distribution of Prime Numbers — Cambridge 30 — 1932
Rendering the set of Cambridge Tracts complete to date.

From Mr. **G. W. Risdon**, the 3rd edition of De Morgan's *Arithmetic*.

From Mr. **F. Robbins**:

E. W. BROWN and D. BROUWER — Tables for the Development of the Disturbing Function - - - - - - - - - 1933

From Mr. **W. Stott**:

G. BOOLE — Differential Equations. Supplementary Volume - 1865
Preparing a posthumous edition of Boole's *Treatise*, I. Todhunter issued the material found among the author's manuscripts in this separate volume rather than attempt the rewriting which incorporation would have involved.

O. FISHER — Physics of the Earth's Crust (2) - - - - - 1889

G. ROSMANITH — Mathematische Statistik der Personenversicherung - 1930

From Mr. **H. E. Tester**, three pamphlets, including

H. E. SOPER — Frequency Arrays - - - - - - - - 1922

The following have been acquired by exchange:

Through Prof. **R. C. Archibald**, a collection of American school books, by C. H. Ashton and W. R. Marsh, W. A. Glanville, J. E. Oliver, L. A. Wait and G. W. Jones, E. Olney (2), H. E. Slaught and N. J. Lennes, E. R. Smith, J. M. Taylor, W. Wells (2), G. A. Wentworth, G. A. Wentworth and G. A. Hill, and G. A. Wentworth, J. A. McLellan and J. C. Glashan.

From the University, Buenos Aires, as a publication exchanged for the *Gazette*:

F. SEVERI — Topologia - - - - - - - - - - 1931
Lectures given at Buenos Aires in 1930.

The following have been bought:

W. H. DREW — Geometrical Conic Sections - - - - - - 1857

P. DE FERMAT — Œuvres: Supplément - - - - - - - 1922
Completing the set.

G. HAGEN — Wahrscheinlichkeits-Rechnung (3) - - - - 1882
The Supplement, *Der Constanten wahrscheinliche Fehler*, 1884, is bound in with this copy. Keynes quotes the author's insistence that to reject observations for no other reason than their discordance with the mean is sheer forgery.

A. R. MAUDUIT — Introduction aux Sections Coniques - - - - 1761
An interesting addition to the collection of books on geometrical conics; in a letter to the Librarian, Mr. Milne says "Considering the date, I think it is a very original work, and for its size, very comprehensive."

R. STEELL — Conic Sections (2) - - - - - - - - 1745
The text is reprinted; reversing a common practice, this edition is 4to, although the first was 8vo, but the paper and print of the first are much superior.

J. M. WILLIAMS — Elements of Euclid I-VI, XI, XII (2) - - - - 1829
"Chiefly from the text of Dr. Simson: with the planes in the eleventh book shaded. Designed, by means of symbols, &c. to facilitate the acquirement of this branch of science."

ERRATA.

VOL. XVII, p. 48, l. 25. *For* 52, 28, 80 *read* 52, 28, 100.

p. 126, l. 5 up. *For* $\frac{\pi}{3}+\gamma-(\alpha+\beta)$ *read* $\frac{\pi}{3}+\gamma-(\beta+\gamma)$.

p. 245, l. 10. *For* $CY=BZ$ *read* $BY=CZ$.

LONDON BRANCH.

PROGRAMME FOR 1933-4.

1933.

Dec. 9th. Presidential Address: "The Finite and the Infinite."—Professor S. BRODETSKY.

1934.

Feb. 3rd. Annual Business Meeting, followed by discussion of Members' Topics. Members are invited to suggest to either Secretary topics they would like discussed.

Feb. 24th. "Some Ideas on Momentum and Energy."—Mr. H. E. PIGGOTT.

Mar. 17th. Discussion of the *Report on the Teaching of Arithmetic in Schools*. Members willing to speak at this meeting are asked to communicate with either Secretary.

A Summer Excursion may be arranged in May or June.

F. A. YELDHAM, C. T. DALTRY } (Hon. Secs.).

SOUTH-WEST WALES BRANCH.

A NEW branch of the Mathematical Association was formed as the result of a meeting at the Swansea Grammar School on Feb. 4th, 1933, of teachers of mathematics from secondary and technical schools in Swansea and district and from the Swansea University College. The following branch officials were elected for 1933: *President*, Prof. A. R. Richardson; *Chairman*, Mr. J. Grey Morgans; *Treasurer*, Mr. G. F. Trubridge; *Secretary*, Mr. T. G. Foulkes. Two further meetings have since been held, both at the Technical College, Swansea.

On March 18th, 1933, Rev. T. H. Ward Hill, of Llandovery College, gave a paper on "The Teaching of Elementary Geometry". After an account of the changes that have occurred in the teaching of geometry during the present century, Mr. Hill dealt with the course up to school certificate stage and described in some detail his own classroom procedure and made many interesting suggestions concerning the grouping of theorems and the teaching of riders. Some reference, not without criticism, was made to the Association Report and the paper closed with some interesting remarks on marking, examination questions and text-books.

The branch held its Summer Meeting on May 27th, 1933, when Professor Richardson gave his presidential address: "The Mathematical Specialist in the School". The speaker illustrated with some powerful examples how some knowledge of higher mathematics can be of assistance to the teacher in his choice of methods, and contended that at all stages of the school the teaching should be in the hands of persons whose knowledge of the subject extended considerably beyond merely what they had to teach.

A pleasing feature of both meetings was the amount of discussion which followed the reading of the papers. The new branch is proving very popular, and its membership is 42. Two further meetings arranged for 1933 are: Nov. 18th, Mr. J. G. Morgans, on "Elementary Algebra and Calculus to School Certificate Stage", and, Dec. 2nd, Dr. W. G. Howell, on "Some Distinctive Landmarks in the Development of Mathematical Symbolism".

T. G. FOULKES (Hon. Sec.).

BOOKS RECEIVED FOR REVIEW.

H. F. Baker. *Principles of Geometry. V. Analytical principles of the theory of curves.* Pp. x, 247. 15s. 1933. (Cambridge)

K. Bartel. *Kotierte Projektionen.* Pp. vi, 80. RM. 4.60. 1933. (Teubner)

W. G. Borchardt. *A first course in mechanics.* Pp. viii, 230, xx. *A second course in mechanics.* Pp. viii, 231-402, xiv. With answers 3s. each. 1933. (Rivingtons)

E. W. Brown and **C. A. Shook.** *Planetary Theory.* Pp. xii, 302. 15s. 1933. (Cambridge)

L. J. Comrie. *The Hollerith and Powers tabulating machines.* Pp. 48. 2s. 1933. (London)

C. V. Durell and **A. W. Siddons.** *Graph Book.* Revised edition. Pp. 66, xiv. 1s. 6d. Boards 1s. 9d. Or in two parts, 10d. each. 1933. (Bell)

L. P. Eisenhart. *Continuous groups of transformations.* Pp. ix, 301. 19s. 6d. 1933. (Princeton University Press; Oxford University Press)

R. Estève et **H. Mitault.** *Cours d'Algèbre. I. Le Calcul algébrique.* Pp. 120. 15 fr. *II. Le Premier degré.* Pp. 152. 15 fr. *III. Le Second degré.* Pp. 172. 15 fr. *Compléments d'Algèbre.* Pp. 76. 10 fr. 1933. (Gauthier-Villars)

H. Galbrun. *Théorie mathématique de l'Assurance invalidité et de l'Assurance nuptialité. Calcul des primes et des réserves.* Pp. viii, 183. 45 fr. 1933. E. Borel's *Traité du Calcul des Probabilités et de ses Applications*, Tome III, Fascicule 5. (Gauthier-Villars)

W. Lietzmann. *Kegelschnittlehre.* Pp. iv, 46. RM. 1.20. 1933. Mathematisch-Physikalische Bibliothek, Reihe I, Band 79. (Teubner)

M. Lindow. *Integralrechnung.* 4. Aufl. Pp. 102. RM. 2.40. 1933. Mathematisch-Physikalische Bibliothek, Reihe II, Band 3. (Teubner)

M. Lindow. *Gewöhnliche Differentialgleichungen.* Pp. vi, 121. RM. 3. 1933. Mathematisch-Physikalische Bibliothek, Reihe II, Band 4. (Teubner)

C. A. Mace. *The Principles of Logic.* Pp. xiii, 388. 12s. 6d. 1933. (Longmans)

H. von Mangoldt und **K. Knopp.** *Einführung in die höhere Mathematik. III.* 6. Aufl. Pp. xvi, 618. Geh. RM. 15. Geb. RM. 16.80. 1933. (Hirzel, Leipzig)

F. Morley and **F. V. Morley.** *Inversive geometry.* Pp. ix, 273. 16s. 1933. (Bell)

N. F. Mott and **H. S. W. Massey.** *The theory of atomic collisions.* Pp. xv, 283. 17s. 6d. 1933. International series of monographs on Physics, 6. (Oxford)

A. S. Percival. *Mathematical facts and formulae.* Pp. vi, 125. 4s. 6d. 1933. (Blackie)

E. A. Price. *Logical sequence in plane geometry. Rapid revision by means of ideograms.* Pp. 12. 1s. 6d. 1933. (Cambridge)

R. Rothe. *Höhere Mathematik für Mathematiker, Physiker und Ingenieure.* Teil IV, Heft 2. Pp. 53-104. RM. 2. 1933. Teubners mathematische Leitfäden, 34. (Teubner)

A. W. Siddons and **C. T. Daltry.** *Elementary algebra. I.* Pp. xxii, 132. 2s. 3d. Without answers, 2s. 1933. (Cambridge)

G. Temple and **W. G. Bickley.** *Rayleigh's Principle and its applications to engineering.* Pp. xiii, 152. 14s. 1933. (Oxford)

A. Véronnet. *Le Calcul vectoriel. Cours d'Algèbre de Mathématiques spéciales et de Mathématiques générales.* Pp. xviii, 252. 50 fr. 1933. (Gauthier-Villars)

E. J. Willis. *Spherical analytic geometry.* Pp. 45. N.p. 1933. (William Byrd Press, Richmond, Virginia; Brown, Son and Ferguson, Glasgow)

A. Wisdom. *Century sum books. III A, III B.* Pp. 48 each. Paper 7d., limp cloth 8d. each. *Teacher's book. III A, III B.* Pp. 48 each. 1s. 3d. each. 1933. (University of London Press)

Mathematical Tables. III. Minimum decompositions into fifth powers. 10s. 1933. Prepared by Professor L. E. Dickson. (Office of the British Association, Burlington House, London)

0.5 Math.

THE MATHEMATICAL GAZETTE

JAN 10 1934

EDITED BY

W. A. A. BROADBENT, M.A.

LONDON

G. BELL & SONS, LTD., PORTUGAL STREET, KINGSWAY, W.C. 2

Vol. XVII., No. 226. DECEMBER, 1933. 2s. Net.

CONTENTS.

	Page
Mathematics at the British Association, 1933. J. H. C. Whitehead.	289
The Envelope of the Simson Lines of a Triangle. A. Lodge.	291
A Proof of the Binomial Theorem. S. Barnard.	294
Games of Chance. I. The Banker's Clock. S. T. Shovelton.	295
The Question of the Momentum. C. H. Chattan-Guinness.	298
The Absolute Summability of Fourier Series. L. S. Bosanquet.	300
Abelian Integrals attached to Algebraic Varieties. W. V. D. Hodge.	305
The Teaching of Geometry. E. H. Neville.	307
Problems connected with a Regular Polygon of N Sides. J. Thayer.	313
Note on the Nature of Trigonometry. V. Naylor.	316
Mathematical Notes (1088–1096). M. F. Egan; A. Ingold; S. Isman; E. H. Neville; C. O. Paradise; J. W. Somers.	319
Reviews. E. J. Atkinson; W. N. Bailey; W. E. H. Berwick; F. C. Boon; R. Cooper; T. M. A. Cooper; H. G. Forder; J. E. Fothergham; N. M. Gibbins; S. Goldstein; C. E. Kemp; E. H. Lockwood; W. H. McCrea; E. H. Neville; G. Pólya; J. Prescott; S. W. Reaveley; A. Robson; W. Stott; E. C. Street; L. G. Vedy; J. Wishart; D. M. Wrinch.	320
Gleanings for and Near (946–960).	325
Index.	xvii–xx

The Mathematical Association.

THE ANNUAL MEETING will be held at the Institute of Education, Southampton Row, London, W.C. 1, on *Thursday, 4th January*, 1934, at 2.30 p.m., and *Friday, 5th January*, 1934, at 10 a.m. and 2.15 p.m.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 10s. per annum, and is due on 1st Jan. It includes the subscription to "The Mathematical Gazette."

Change of Address should be notified to G. Pendlebury, M.A., 29, Burlington Road, Chiswick, W.4. If Copies of the "Gazette" fail for lack of such notification to reach a member, duplicate copies can be supplied only at the published price.

Subscriptions should be paid to Mr. W. H. Jay, 68 Marlborough Road, Chiswick, London, W.4.

G. BELL & SONS

ELEMENTARY CALCULUS

By C. V. DURELL, M.A., *and* A. ROBSON, M.A.

Volume I covers all that is required for "additional" mathematics and mathematics "more advanced" in Certificate and Matriculation examinations. Volume II deals comprehensively with Higher Certificate work, including distinction papers. [The sections of Volume II required for ordinary Higher Certificate requirements are to be issued separately.] **Both volumes are issued with or without an Appendix of additional examples.**

"Definitely an important book, not only because of the established reputation of its authors, but because it contains new and attractive features."—JOURNAL OF EDUCATION

Vol. I, **3s. 6d.**; *with Appendix*, **4s. 6d.** *Vol. II immediately.*

ELEMENTARY TREATISE ON PURE MATHEMATICS

By N. R. CULMORE DOCKERAY, M.A.

MATHEMATICAL MASTER, HARROW SCHOOL

This new book provides, in one comparatively inexpensive volume, a comprehensive course of elementary analysis suitable for scholarship candidates in schools and for University students. It endeavours to combine rigour with simplicity and clear exposition. A feature is the large number of examples, of which there are about 1,500.

Ready shortly. *Approx.* 600 *pages.* *About* **15s.** *net.*

G. BELL & SONS, LTD., PORTUGAL ST., W.C. 2

Just published

INVERSIVE GEOMETRY

By

FRANK MORLEY, M.A., Sc.D. (Camb.)
PROFESSOR-EMERITUS OF THE JOHNS HOPKINS UNIVERSITY

and F. V. MORLEY, D.Phil. (Oxford)

Price* 16s. *net.

An introduction to algebraic geometry with special reference to the operation of inversion, showing the use of modefn views in comparatively elementary and practical questions. The book is divided into two parts.

In PART I the authors go over the application of number to Euclid's planar geometry, and arrive at the Euclidean group of operations. By adjoining an inversion they treat the wider inversive group. Klein's views of a geometry are applied to the inversive group of a plane. A chapter follows on Flow which may serve to introduce matter, and a chapter is added on Differential Geometry. PART II is composed of applications. Some are of the most elementary nature, but the authors believe that the tradition that simple geometric and mechanical questions are to be handled only as Euclid or Descartes might have handled them is very hampering.

PROSPECTUS ON REQUEST

G. BELL & SONS, LTD., PORTUGAL ST., W.C.2

Cambridge University Press

PRINCIPLES OF GEOMETRY

By H. F. BAKER.

Volume V—Analytical Principles of the Theory of Curves

Demy 8*vo.* 15*s. net.*

This Volume, and Volume VI, which is in the press, have the purpose of carrying the reader to the threshold of the modern invariantive theory of surfaces and algebraic loci in general. They contain a summary of the fundamental algebraic and functional notions which experience has shown to be requisite for a proper comprehension of this general theory. Volume V in particular is an analytical account of the *Theory of Curves* and the associated algebraic integrals, with the elements of the theory of Riemann surfaces, the consideration of Correspondence being, however, postponed to Volume VI.

PLANETARY THEORY

By E. W. BROWN and C. A. SHOOK. *Demy* 8*vo.* 15*s. net.*

This volume consists mainly of methods for the calculation of the action of one planet on another. While the earlier chapters on elliptic motion and on the development of the disturbing forces are applicable to all the known methods of treatment of the problem, the later chapters deal in detail with those which use the variations of the elements and with that in which the true longitude is used as the independent variable. Special attention is directed to the second approximations in powers of the masses. The general theory of resonance is also developed, and the final chapter is devoted to the Trojan groups of asteroids.

TABLES FOR THE DEVELOPMENT OF THE DISTURBING FUNCTION

With Schedules for Harmonic Analysis

By E. W. BROWN and D. BROUWER. *Royal* 4*to.* 10*s.* 6*d. net.*

AN INTRODUCTORY COURSE OF MECHANICS

By E. G. PHILLIPS. *Demy* 8*vo. With* 101 *text-figures.* 10*s.* 6*d. net.*

This book does not presuppose any previous acquaintance with the subject, which the author regards throughout as a branch of Mathematics, not as an ancillary to Physics or Engineering. The content of the book covers the scope of a one-year's course in Mechanics.

SCHOOL CERTIFICATE MATHEMATICS

By H. J. LARCOMBE and J. K. FLETCHER.

Geometry, Part I

Crown 8*vo.* 2*s. With answers,* 2*s.* 3*d.*

In this series the authors will provide graded courses in Mathematics based on the syllabuses of the various Examining Bodies and on the suggestions contained in the recently published report on School Certificate examinations. The Algebra and Geometry series, which are in course of preparation, will consist each of three parts.

LOGICAL SEQUENCE IN PLANE GEOMETRY

Rapid Revision by means of Ideograms

By E. A. PRICE.

Demy 8*vo. With* 1 *folded sheet of Ideograms.* 1*s.* 6*d.*
Folded sheets of Ideograms separately in envelope, 6*d. a dozen.*